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Solving loop equations by Hitchin systems via holography in large-N QCD_4

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Abstract: For (planar) closed self-avoiding loops we construct a "holographic" map from the loop equations of large-N QCD_4 to an effective action defined over infinite rank Hitchin bundles. We compute the effective action in a closed form in terms of functional determinants. The construction works as follows. Large-N QCD_4 on $T^2 \times T^2$ ($T^2 \times T^2_\theta$) is equivalent by Eguchi-Kawai to a partially quenched (twisted) theory on T^2 described non-canonically by Hitchin pairs ($A = A_z dz + A_{\bar{z}} d\bar{z}, \Psi = -iD_u dz + i\bar{D_u} d\bar{z}$) with A transforming as a connection and Ψ as a Higgs field with respect to the gauge group on T^2. (Centrally extended) parabolic Hitchin bundles classify a dense set of orbits of pairs ($A, \Psi = \psi + \bar{\psi}$) obtained as hyper-Kahler quotient by the action of the gauge group. We study the loop equations of the quenched (twisted) theory for the non-hermitean connection $B = A + i(\kappa\psi + \bar{\kappa}\bar{\psi}); \kappa \neq 0$ and planar loops. We change variables in the loop equations from (centrally extended) parabolic Hitchin bundles to the corresponding (centrally extended) holomorphic (i.e. chiral) de Rham local systems. The key point is that the quantum contribution to the loop equations for the holomorphic de Rham local systems can be evaluated by residues in a certain regularization and turns out to depend on the loop orientation and self-intersection. The outcome is that, for self-avoiding loops, the original loop equations are implied by the critical equation of an effective action (the master equation) defined over the microcanonical ensemble of the inductive limit of finite dimensional gauge orbits in the quenched theory, or over infinite dimensional orbits, arising from a central extension of Hitchin bundles, in the twisted one. Finally, on the basis of the geometric Langlands correspondence, we propose a dual construction in which the moduli of Hitchin bundles are quantised and become differential operators. The loop equations are then implied by the condition that the master equation annihilates the vacuum and the master field is realized as an element of a quasi-conformal (vertex) algebra.

Keywords: Renormalization Group, 1/N Expansion, Differential and Algebraic Geometry, Non-Commutative Geometry.

Contents

1. Introduction

To say it in a nutshell, in this paper we construct an effective action whose critical points solve the loop equations [1, 2] of large-N QCD in four dimensions [3] for (planar) closed self-avoiding loops.

Being a functional over Hitchin systems [4, 5] (see appendix B), our effective action is, in a sense, the analogue for large-N QCD_4 of the Seiberg-Witten low energy effective action for the abelian branch of $\mathcal{N} = 2$ $SUSY$ gauge theories in four dimensions [6]. However, we should stress that our effective action is derived directly by the loop equations of the large-N pure gauge theory in four dimensions for (planar) closed self-avoiding loops, once carefully chosen variables are employed in the functional integral.

The effective action is derived via a "holographic" map [7]–[9], that allows us to map the loop equations, that live on the boundary, i.e. on the loops, into a master equation, that is the critical equation of an effective action defined on the bulk, i.e. on the space-time.

One key feature responsible for the existence of the "holographic" map is a "chiral" change of variables, that allows us to reduce the computation of the quantum contribution in the loop equations to the evaluation of a residue, that depends only on the loop orientation for self-avoiding loops.

Another important feature of our "holographic" map is that the residue, that arises from the contour integral along planar self-avoiding loops in the loop equations, is cancelled by a $u(1)$ anomalous contribution from the "chiral" change of variables in the bulk, provided the conformal factor of the metric is suitably chosen.

Despite the holographic suggestions, in this introduction we would like to make contact with the concept of master field [10], that would occur as the solution of the master equation for our effective action, with special regard to some algebraic features that arose in the subject.

From a purely algebraic point of view, it is of the utmost importance that the algebra generated by the monodromy of the connection that solves the loop equations for closed self-avoiding loops be hyper-finite, i.e. be the weak limit of a sequence of finite dimensional von Neumann algebras (see appendix A), a property shared (trivially) by the algebra generated by the distribution of the eigenvalues that solves the large-N limit of some matrix models [11]. For the existence of an effective action defined over a hyper-finite algebra (a HF effective action in short) a crucial role is played by the condition that closed loops be self-avoiding in the loop equations. Indeed it has been known for some time that in the general case of arbitrary self-intersection, the ambient algebra, in which the connection that solves the loop equations lives, is (algebraically isomorphic to) a free group factor with an infinite number of generators (see appendix A), that is not a hyper-finite von Neumann algebra. The lack of hyper-finiteness seems to be one of the reasons that so far prevented an exact treatment of the general case.

The existence in large-N QCD_4 of a hyper-finite effective action for (planar) closed self-avoiding loops, as opposed to the general case, has a precise analogue in large-N QCD_2 that is explicitly solvable [12]. In the two dimensional case the von Neumann algebra generated by non-overlapping but self-intersecting loops at a point is precisely the free group factor with an infinite number of generators [13]–[16], a type II_1 non-hyper finite von Neumann algebra [16]. However functional techniques show that the master field, that solves the loop equations of large-N QCD_2 for closed self-avoiding loops, lives instead inside a commutative hyper-finite von Neumann algebra and indeed it can be determined by the distribution of the eigenvalues [17]. This fact suggested to the author to look for hyper-finite solutions of the loop equations for closed self-avoiding loops in the four dimensional case. In a sense this approach follows the original Witten proposal for the master field [10], that contained implicitly some kind of finite dimensional approximation property in the prescription of evaluating gauge invariants operators on the master field by means of the normalised approximately finite dimensional matrix trace $\lim_N \frac{1}{N}\,\mathrm{Tr}_N$. In fact the existence of an approximately finite dimensional trace is weaker than hyper-finiteness but implies that the von Neumann algebra of the corresponding GNS representation is embeddable in the factor R^ω [18], the ultra power of the unique hyper-finite factor R (see appendix A).

The plan of the paper is the following one. Section 2 is an algebraic detour in which we recall some old and new ideas about the master field. Though it is not strictly necessary for the rest of the paper it helps to put section 3 and section 4 in the proper algebraic perspective. We make also some comment on the physical meaning of restricting to closed self-avoiding loops. In addition in section 2 we recall the quenched [19, 20] and twisted [21]

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Eguchi-Kawai (EK) [19] reductions that are known to be equivalent to the large-N limit of QCD. The quenched and twisted theories that we refer to are the continuum version of their original lattice formulation [19, 22, 23]. A partially quenched [24] or twisted theory [21] is used in this paper as a technical tool to embed an inductive sequence of finite dimensional Hitchin systems into the QCD_4 functional integral.

Section 3 contains a first change of variables, from the gauge connection to the moduli fields of finite dimensional gauge orbits obtained by hyper-Kahler [4, 25] reduction in a partially quenched Eguchi-Kawai model or to the moduli fields of the hyper-Kahler reduction of a certain infinite dimensional central extension of Hitchin systems in the twisted case. In this section the functional integral is represented as an integral over the (energy) levels of the microcanonical ensemble for the action of the local gauge group. However an effective action on the gauge orbits, that turn out to be Hitchin systems (or an infinite dimensional central extension of, in the twisted case), cannot be derived by the loop equations for these variables yet.

In section 4 the effective action is constructed by means of a further change of variables, from the moduli fields of the orbits of the SU(N) gauge group to the moduli fields of the corresponding holomorphic (see appendix B) de Rham local system (or a central extension of).

Objects related to the holomorphic de Rham local system have been introduced under the name of opers by Beilinson and Drinfeld in their study of the geometric Langlands correspondence [26](opers will play a role in section 6, after quantising Hitchin systems as an alternative tool to solve the loop equations). The construction of the effective action involves a de Rham local system for a partially quenched or twisted Eguchi-Kawai model. Hyper-finiteness is shown essentially by construction in the quenched version, while it follows with little more effort in the twisted theory, since in this case the gauge algebra has to be defined from start on an infinite dimensional Hilbert space and only later on approximated by compressions that define finite dimensional representations of the fundamental group. In any case hyper-finiteness should hold also in the twisted theory, at least because of the equivalence of the quenched and twisted theories in the large-N limit.

In section 5 the effective action is computed in terms of functional determinants. The effective action is the sum of three terms. The first one is the classical action of the EK partially quenched or twisted model. The second one is the logarithm of a functional determinant that arises from localisation on the microcanonical ensemble. The third one is the logarithm of an "anomalous" jacobian that arises from the change of variables from the moduli of Hitchin systems to the moduli of the holomorphic (i.e. chiral) de Rham local systems. Despite the techniques employed in this paper are mostly based on two dimensional differential geometry, we find that the localisation determinant on the microcanonical ensemble of the Hitchin systems can be computed in a manifestly four dimensional euclidean (or Lorentz) invariant way. Remarkably we find that the first coefficient of the beta function is exactly reproduced, at lowest order in the expansion in powers of the moduli fields (that at lowest order coincides with the usual perturbative expansion), by the contribution of the localisation determinant in the master equation. It follows immediately that the techniques employed here apply also to the $\mathcal{N}=1$ super-symmetric extension of large-N

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QCD_4 obtained adding to the pure gauge theory minimally coupled Majorana gluinos in the adjoint representation, since in this case the $\mathcal{N}=1$ effective action is obtained simply adding the logarithm of the fermion pfaffian to the effective action of the pure gauge theory.

In section 6, using ideas based on the geometric Langlands correspondence [26]–[32] (and references therein), we suggest another way of solving loop equations for closed self-avoiding loops by quantising Hitchin systems. In this last formulation, due to quantisation, hyper-finiteness of the ambient algebra may get lost, keeping nevertheless (quantum) integrability.

In this case the master field becomes an element of a quasi-conformal (vertex) algebra. This is very close to interpreting the master field as an element of a vertex algebra of a conformal field theory, as it would be required by the conjectured gauge-field/string duality [33, 34], but for the fact that our algebra is obtained first solving the Hitchin self-duality equations on a punctured torus and then quantising the resulting moduli space, while in the usual construction of conformal blocks first the space of two dimensional connections is quantised and then the quantised Hitchin equations are imposed as constraints on the space of physical states [35, 36].

Our conclusions are contained in section 7.

In appendix A we recall some elementary facts about von Neumann algebras, that may help to understand the revival of the algebraic approach to the master field.

In appendix B we remind the reader of several notions of geometrical nature, about Hitchin bundles, that are used in the paper.

In appendix C we compare our approach to the loop equations with Witten localisation formulas in QCD_2 [37]–[40].

In appendix D we compute, in a manifestly euclidean (or Lorentz) invariant way, the divergences of the localisation determinant at lowest order in the expansion in powers of the moduli fields.

2. Loop equations: an algebraic detour

Recently there has been a revival of applications of ideas in the theory of von Neumann algebras (for a comprehensive review see [41]) to field theory, in particular to the large-N limit (see for a recent review [42]). We regard the large-N limit as defined by the loop equations [1, 2], that are the Schwinger-Dyson equations in the large-N limit for the expectation values of traces of Wilson loops in gauge theories or for a complete set of gauge invariant operators in other theories (i.e. matrix models). They are obtained noticing that the integral of a derivative vanishes:

$$\begin{aligned} 0 &= \int DA_\mu \operatorname{Tr} \frac{\delta}{\delta A_\nu(z)} \left(\exp\left(-\frac{N}{4\lambda} \int \operatorname{Tr} F_{\mu\nu}^2 d^4x \right) \Psi(x,x;A) \right) \\ &= \int DA_\mu \exp\left(-\frac{N}{4\lambda} \int \operatorname{Tr} F_{\mu\nu}^2 d^4x \right) \times \\ &\quad \times \left(\frac{N}{\lambda} \operatorname{Tr}(D_\mu F_{\mu\nu}(z)\Psi(x,x;A)) + i \int_{C(x,x)} dy_\nu \delta^{(d)}(z-y) \operatorname{Tr}(\lambda^a \Psi(x,y;A)\lambda^a \Psi(y,x;A)) \right), \end{aligned} \tag{2.1}$$

where the sum over the index a is understood. Here λ^a are hermitean generators of the Lie algebra and

$$\Psi(x,y;A) = P\exp i\int_{C_{(x,y)}} A_\mu dx_\mu \tag{2.2}$$

$\Psi(x,x;A)$ is the monodromy matrix of the connection A_μ along the closed loop $C(x,x)$ based at the point x, i.e. $\Psi(x,x;A)$ is the Wilson loop. For the group $\mathrm{SU}(N)$ using the identity:

$$\lambda^a_{\alpha\beta}\lambda^a_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta} \tag{2.3}$$

we get:

$$\begin{aligned}0 = &\int DA_\mu \exp\left(-\frac{N}{4\lambda}\int \mathrm{Tr}\, F^2_{\mu\nu} d^4x\right)\times\\ &\times\Bigg(\frac{N}{\lambda}\,\mathrm{Tr}(D_\mu F_{\mu\nu}(z)\Psi(x,x;A)) + \\ &\quad + i\int_{C(x,x)} dy_\nu \delta^{(d)}(z-y)\left(\mathrm{Tr}(\Psi(x,y;A))\mathrm{Tr}(\Psi(y,x;A)) - \frac{1}{N}\mathrm{Tr}(\Psi(x,y;A)\Psi(y,x;A))\right)\Bigg),\end{aligned} \tag{2.4}$$

where the last term may be disregarded in the large-N limit. The second term is the quantum contribution to loop equations, while the first term is the classical one. Thus, using the factorisation of gauge invariant operators in the large-N limit and noticing that the expectation value can be combined with the matrix trace to define a new generalised trace τ, our problem is to find a (unique) solution $A_\mu(x)$ to

$$\begin{aligned}0 = &\frac{1}{\lambda}\tau(D_\mu F_{\mu\nu}(z)\Psi(x,x;A)) + \\ &+ i\int_{C(x,x)} dy_\nu \delta^{(d)}(z-y)\tau(\Psi(x,y;A))\tau(\Psi(y,x;A))\end{aligned} \tag{2.5}$$

for every closed contour C, with values in a certain operator algebra with normalised ($\tau(1) = 1$) trace τ [13]–[15]. Such a solution is named the master field [10]. The recent revival concerns the precise identification of the algebra and the trace relevant for the large-N limit of field theories [13]–[15]. Operator algebras have already played some role in the early days of constructive field theory, but they have been later abandoned in favour of functional integral techniques as soon as it became clear that the algebra of a single euclidean scalar field is commutative. In our case however the algebra of euclidean matrix fields in the large-N limit is highly non commutative.

The recent interest in the interplay between the theory of von Neumann algebras and the large-N limit was driven on the mathematical side by Voiculescu use [16] of large-N matrix models as a tool in the study of non hyper-finite von Neumann algebras, in particular of the "elusive" free group factors that are left out by Connes [41] algebraic classification of all hyper-finite factors (see appendix A). On the physical side, after Singer suggestion [44] that type II_1 representations of Voiculescu free group factors are the relevant ones for the study of the large-N limit, several old and new results about the master field have been interpreted in the light of the theory of von Neumann algebras [13]–[15]. These investigations culminated with the precise identification of the ambient algebra and the trace for all matrix models in the large-N limit, at least to all orders in the planar perturbation theory [13]–[15]. In this section we recall some old and recent ideas about the master field, interpreting them from the more recent algebraic point of view (see [42] for a review). Though this is possibly historically incorrect, it is useful for our purposes. After the discovery of large-N factorisation [43] of gauge invariant local operators, Witten suggested that the large-N limit could be understood as localisation of the functional integral on a unique gauge orbit, the master orbit, on which gauge invariant operators should be evaluated by means of the usual normalised matrix trace in the large-N limit [10]. In the retrospect we observe that the use of this matrix trace is tantamount to postulating that the von Neumann algebra generated by the master field is of type II_1 [44] and admits an approximately finite dimensional trace, in particular is embeddable into R^{ω}, the ultra product of the unique hyper-finite factor R. If in addition the trace is uniformly approximately finite dimensional the associated GNS representation is hyper-finite (HF) (see appendix A) [18]. While the aim of this paper is to find observables in large-N QCD_4 for which the original Witten idea of a possibly hyper-finite master field can be concretely realized, in general the hyper-finiteness condition turned out too much restrictive, even in the case of the large-N limit of a free scalar field [45]. Haan [46] discovered that the von Neumann algebra of the master field for a free scalar field was the Cuntz algebra with an infinite number of generators (though he did not give it this name). Haan found the following representation for the master field of a free scalar field in the large-N limit:

$$\phi(x) = \int a(p) \exp(ipx) + a(p)^* \exp(-ipx) dp \tag{2.6}$$

with the operators $a(p)$ and $a(p)^*$ satisfying:

$$\begin{aligned} a(p)a(q)^* &= \delta^{(d)}(p-q) \\ a(p) \mid \Omega\rangle &= 0\,, \end{aligned} \tag{2.7}$$

where $\mid \Omega\rangle$ is the vacuum state. In addition he found, on the basis of the large-N Schwinger-Dyson equations, that also in the interacting case of a scalar theory with quartic interaction (but in a way that trivially extends to any interaction) the ambient algebra of the master field was the vacuum (i.e. tracial) GNS representation of the Cuntz algebra. The Cuntz algebra in general is generated by the following creation and annihilation operators, that satisfy:

$$a_i a_j^* = \delta_{ij} 1 \tag{2.8}$$

and no other relation but the completeness condition:

$$\sum_i a_i^* a_i = 1 - P_\Omega\,, \tag{2.9}$$

where P_Ω is the orthogonal projector into the vacuum. A tracial representation is obtained requiring:

$$a_i \mid \Omega\rangle = 0\,, \tag{2.10}$$

where $\mid \Omega\rangle$ is the vacuum. After Voiculescu work we know that the algebra generated by the self-adjoint elements $a_i^* + a_i$ is a type II_1 algebra not hyper-finite for $i \geq 2$. The

Cuntz algebra with more than one self-adjoint generator in its tracial representation is algebraically isomorphic to the free group factor with an equal number of generators, that was known already to von Neumann to be an example of a non-HF factor [41]. The occurrence of free group factors in the large-N limit of field theories is implicit in the work of [47] on the effective action of the large-N limit via the introduction of symbols that "neither commute nor anti-commute" (an informal definition of a free algebra) and in a way or another enters (implicitly) most of the subsequent developments (for example in the use of the loop algebra [48, 49]) until it has been made explicit in the work of [13]–[15] through the introduction of the Cuntz algebra as the ambient algebra for the master field.

In particular in [13]–[15] it is proved, to every order in planar perturbation theory, that the master algebra is a sub-algebra of the Cuntz algebra generated by the following elements:

$$M_\mu = a^*_\mu + M_{\mu\nu} a_\nu + M_{\mu\nu\rho} a_\nu a_\rho + \cdots , \tag{2.11}$$

where $M_{\dots}$ are the planar correlation functions, so that the knowledge of the master field is in fact equivalent to the knowledge of the solution of the planar limit. To summarise, we know that the master algebra of a large-N free field is a non-HF Cuntz algebra and that the master algebra of an interacting large-N matrix model is a sub-algebra of the Cuntz algebra, probably non-HF too. However in some very special cases, such as the one-matrix model [11], traces of local operators can be reconstructed by of the distribution of the eigenvalues, that generates an abelian HF algebra (in fact in this case the master algebra is the commutative von Neumann algebra generated by M_μ for just one μ). In this case HF does not come as a surprise, since a commutative C^*-algebra is of type I and therefore HF (see appendix A). We regard the lack of HF in the general case as the most disappointing, making difficult any progress in finding exact solutions of the loop equations (unless, perhaps, some kind of exact integrability is present). In this paper we renounce to attempt a solution of the large-N limit for all observables, but rather we concentrate on those physically relevant observables for which a HF solution may exist. In this search we use as a guide the important example of large-N QCD_2 that is explicitly solvable [12]. For QCD_2 a gauge exists in which the theory is gaussian, so that the master algebra is generated by the Cuntz algebra in that gauge [13]–[15]. In addition the algebra generated by Wilson loops based at a point is the free group factor with an infinite number of generators, one for each simple loop, i.e. non overlapping self-intersecting loop at a base point [14]. This can be proved noticing [14] that the iterative solution of the Migdal-Makeenko equations with respect to the number of self-intersections coincides with the iterative equation associated to a Voiculescu free family of non-commutative random variables [16], i.e.:

$$\langle (W(A_1) - \langle W(A_1)\rangle) \cdots (W(A_n) - \langle W(A_n)\rangle)\rangle = 0 \,. \tag{2.12}$$

This provides iterative equations for the expectation values of products of n random variables in terms of expectation values of a lower number of variables. The entire family is thus determined by the expectation values of self-avoiding loops. For each simple loop of area A_k it is possible to give an explicit unitary master field in terms of the Cuntz algebra [14]:

$$W(A_k) = \exp(i\sqrt{A}_k(a_k + a^*_k)) \,. \tag{2.13}$$

However the expectation value of a closed self-avoiding Wilson loop on a large sphere can be computed by means of the distribution of the eigenvalues and as such admits a master field that generates a commutative HF algebra, as the master field of the one-matrix model. Indeed, on a sphere of large area, the average of a Wilson loop of area A is given by [50]:

$$W(A) = Z^{-1} \int \prod d\theta_i \prod_{i \neq j} (\theta_i - \theta_j) \exp\left(-\frac{N}{2A} \sum_i \theta_i^2\right) \times \sum_i N^{-1} cos\theta_i \,, \tag{2.14}$$

where:

$$Z = \int \prod d\theta_i \prod_{i \neq j} (\theta_i - \theta_j) \exp\left(-\frac{N}{2A} \sum_i \theta_i^2\right). \tag{2.15}$$

In the large-N limit the distribution of the eigenvalues is determined by the saddle-point equation:

$$\frac{N}{2A}\theta_i = \sum_{j \neq i} \frac{1}{(\theta_i - \theta_j)} \,. \tag{2.16}$$

In section 4 we will see that the existence of HF solutions of the loop equations is somehow linked to the condition of non self-intersection. But even in this case, considerable difficulties arise when a “simple” (i.e. not living in a free group algebra) solution of the loop equations is attempted. Indeed for a closed self-avoiding loop based at x the loop equations degenerate into:

$$\begin{aligned} 0 = & \int DA_\mu \exp\left(-\frac{N}{4\lambda} \int \mathrm{Tr}\, F_{\mu\nu}^2 d^4x\right) \times \\ & \times \mathrm{Tr} \left(\left(\frac{N}{\lambda} D_\mu F_{\mu\nu}(x) + i \int_{C(x,x)} dy_\nu \delta^{(d)}(x-y)\, \mathrm{Tr}(1)\right) \Psi(x,x;A)\right). \end{aligned} \tag{2.17}$$

We see therefore that, despite the equations have been ”linearised” in such a way that the classical and quantum terms have the same operator structure, the quantum contribution has still a different functional dependence on the loop C from the classical term, because the distribution:

$$i \int_{C(x,x)} dy_\nu \delta^{(d)}(x-y) \tag{2.18}$$

is obviously loop dependent. This is even more manifest writing the loop equations for self-avoiding loops in the Migdal-Makeenko form [1, 2]:

$$0 = \partial_\mu(x) \frac{\delta \langle \mathrm{Tr}\, \Psi(x,x;C) \rangle}{\delta \sigma_{\mu\nu}(x)} + \int_{C(x,x)} dy_\nu \delta^{(d)}(x-y)\, \mathrm{Tr}(1) \langle \mathrm{Tr}\, \Psi(x,x;C) \rangle \,. \tag{2.19}$$

Somehow anticipating the result of section 4 we will manage to change variables in the loop equations in such a way that in the new variables the corresponding distribution in the quantum contribution will be loop independent for self-avoiding loops. This will open the way to find an equivalent effective action.

Since we want to solve loop equations only for closed self-avoiding loops we wonder how much physical information is lost by this restriction. Obviously we can reconstruct by

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such loops the string tension at large distances and the running of the coupling constant at short distances. The next problem is the spectral information, i.e. the meson or the glueball spectrum. This turns out to be dependent on the dimension of space-time. Essentially, for the meson spectrum, we want to compute the Dirac propagator in the background of the master field. By employing a path integral representation for the propagator, we represent it as an average over Wilson loops computed along brownian paths (for a recent use of this representation in the framework of the large-N limit of QCD see [51, 52]). Now in $d = 2$ brownian paths self-intersect any number of times with probability one, while in $d = 4$ they are self-avoiding with probability one (see for example [53]). Therefore in $d = 4$ the spectral information is contained in self-avoiding loops. An analogous argument holds for the glueball spectrum. We should mention that the construction of the effective action that we present in this paper holds only for planar loops, while in general to reconstruct the meson or the glueball spectrum we would need also non-planar ones. Since, however, every loop can be made to lie on a surface, we think that our construction can be extended to the general case at the price of making the geometry more involved, though this will not be attempted in this paper. We end this section recalling the EK reduction [19], that plays a significant role in our approach to the loop equations. The quenched EK reduction [19, 20] is essentially a way of constructing a theory with the same master field, i.e. the same large-N loop equations, but with fluctuations reduced by a factor of N^{-1}. In the process space-time degrees of freedom are reduced to lower, eventually zero dimensions. The partition function of the quenched EK model is given by:

$$Z = \lim_{N\to\infty} \int dA^{ch}_\mu \exp\left(\frac{N(2\pi)^d}{4\lambda\Lambda^d} \operatorname{Tr}[p_\mu + A^{ch}_\mu, p_\nu + A^{ch}_\nu]^2\right), \tag{2.20}$$

where the integration is now over charged constant $N \times N$ matrices A^{ch}_μ and p_μ are diagonal matrices with uniform distribution of eigenvalues, i.e. p_μ are the quenched momenta. In the Eguchi-Kawai quenched prescription the volume of the phase space divided by $(2\pi)^d$ is assumed to scale as N according to $(2\pi)^{-d}\Lambda^d V = N$ in such a way that the action is effectively rescaled by a factor of N in order to compensate for a reduction of a factor of N in the entropy of the functional integration, that now is only over constant matrices. One way of seeing that the EK model is equivalent to the large-N limit of QCD is to look at its loop equations:

$$\begin{aligned} 0 = \int dA^{ch}_\mu \exp\left(\frac{N(2\pi)^d}{4\lambda\Lambda^d} \operatorname{Tr}[p_\mu + A^{ch}_\mu, p_\nu + A^{ch}_\nu]^2\right) \times \\ \times \left(\frac{N}{\lambda} \operatorname{Tr}\left([p_\mu + A^{ch}_\mu, [p_\mu + A^{ch}_\mu, p_\nu + A_\nu]]\Psi(x,x;A)\right) - \right. \\ \left. - i \int_{C(x,x)} dy_\nu \frac{\Lambda^d}{(2\pi)^d} \operatorname{Tr}(\Psi(x,y;A)) \operatorname{Tr}(\Psi(y,x;A))\right). \end{aligned} \tag{2.21}$$

Provided that the coupling constant was rescaled as in eq. (2.20), these are the same loop equations as for large-N continuum QCD, but for the fact that there is no delta function in the quantum contribution, since in the reduced model we are integrating over constant matrices. Therefore the loop equations of the reduced EK model coincide with the ones of the un-reduced theory provided the trace of open loop vanishes, that is a way of reproducing the delta function. This is ensured by the global R^d symmetry (if it remains unbroken) (see [42]):

$$A_\mu \to p_\mu 1 + A_\mu \tag{2.22}$$

that implies:

$$\frac{1}{N}\langle \mathrm{Tr}(\Psi(x,y;A))\rangle = \exp(ip(x-y))\left\langle \frac{1}{N}\mathrm{Tr}(\Psi(x,y;A))\right\rangle . \tag{2.23}$$

In fact the quenching prescription was introduced precisely for this aim. A variant of the quenched EK model is the twisted model [21]:

$$Z = \int dA_\mu \exp\left(\frac{N}{4\lambda}\mathrm{Tr}[\Gamma_\mu + A_\mu, \Gamma_\nu + A_\nu]^2\right), \tag{2.24}$$

where Γ_μ are (unbounded) self-adjoint operators acting on an infinite dimensional Hilbert space satisfying:

$$[\Gamma_\mu, \Gamma_\nu] = i(\theta^{-1})_{\mu\nu}1 \tag{2.25}$$

with $(\theta^{-1})_{\mu\nu}$ vanishing in the large-N limit. In modern language the twisted model is a gauge theory defined over the non-commutative euclidean space:

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}1 . \tag{2.26}$$

The non-commutative Yang-Mills theory (for a review see [54]) equivalent to the twisted model is obtained working with commuting coordinates and replacing the ordinary product between functions with the Moyal star product defined by:

$$f(x)\star g(x) = \exp\left(i\theta_{\mu\nu}\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu}\right) f(x+y)g(x+z)\ |_{y=z=0} \ . \tag{2.27}$$

Thus the partition function of the twisted theory can be re-casted into its non-commutative form, that looks like a matrix model:

$$Z = \int dA_\mu \exp\left(-\frac{N}{4\lambda}\mathrm{Tr}(i[C_\mu, C_\nu] - (\theta^{-1})_{\mu\nu})^2\right), \tag{2.28}$$

where $C_\mu = (-i\theta^{-1})_{\mu\nu}x_\nu + iA_\mu$ and the product is the Moyal product. The loop equations of the non-commutative theory [55, 56] differ from those of the commutative theory for the need of taking an integral over the base-point of the Wilson loop, in order to get gauge invariant quantities, since the trace now includes integration over the base point of the Wilson loop. However, after taking the expectation value, the trace of Wilson loops is proportional to the volume of space-time, because of translational invariance, so that we get a normalised trace dividing by the volume. In the $\theta_{\mu\nu} \to \infty$ limit the loop equations of the non-commutative theory (for any positive N, even $N = 1$) coincide with the ones of the large-N commutative theory.

3. The microcanonical ensemble

It is quite well known that every euclidean field theory can be thought as the partition function of a system of classical statistical mechanics in which the coupling constant squared is identified with the classical temperature (see for example [57]). Following this interpretation the euclidean functional integral:

$$\int \exp\left(-\frac{1}{\lambda}S(A)\right) DA \tag{3.1}$$

looks like the classical partition function:

$$\int \exp(-\beta H(p,q))dpdq\,. \tag{3.2}$$

The second integral has an obvious representation as an integral over the energy levels of the microcanonical ensemble (see for example [58]):

$$\begin{aligned}\int \exp(-\beta H(p,q))dpdq &= \int \exp(-\beta E)\delta(H(p,q)-E)dpdqdE \\ &= \int \exp(-\beta E)\,\mathrm{Det}(\nabla H)^{-1}dE\,.\end{aligned} \tag{3.3}$$

The aim of this section is to find an analogous representation for the partition function of QCD_4. The motivation is the following one. The original Witten proposal for the master field can be interpreted as a sort of localisation of the functional integral at large-N on a certain microcanonical ensemble specified by the set of parameters that describe the master gauge orbit. It is therefore natural to classify explicitly gauge orbits. According to the discussion in section 2 we assume that the relevant gauge orbits are those of hyper-finite type, i.e. those which can be obtained as limits of finite dimensional orbits. In section 4 we will inquire whether and to what extent localisation on this microcanonical ensemble is consistent with the loop equations. In this section we simply define the microcanonical ensemble. Following our microcanonical analogy it is clear that the role of the energy should be played by a set of levels of the hamiltonians that parameterise the gauge orbits. The levels will be therefore part of the moduli of the orbits. In addition, since the gauge group acts locally, the moduli will be moduli fields. The representation that we are going to find is an identity as it is eq. (3.3) and there is a priori no guarantee that it will help us to derive an effective action defined over the microcanonical ensemble from the loop equations. Indeed it is still necessary to functionally integrate over the moduli fields, whose fluctuations a priori will contribute loop dependent quantum terms in the loop equations, as it happens for the original integration variables, i.e. the gauge connection. In the next section we will show that an effective action exists for (planar) closed self-avoiding loops, provided we make a further change of variables from the levels of the hamiltonians to the moduli of the associated holomorphic de Rham local system and we allow a central extension in the defining equations of the microcanonical ensemble. The existence of the central extension does not spoil the hyper-finiteness property in the case of the partially quenched Eguchi-Kawai model, but HF is less obvious in the twisted theory since in this case gauge orbits have to be infinite dimensional from start. However the HF property should be kept also in this case, since the quenched and twisted models are equivalent in the large-N limit, though we will not provide any direct proof of it. Following our statistical-mechanical analogy we want to think the euclidean functional integral as an integral over the phase space rather than the configuration space. DA should be therefore a Liouville measure associated to some symplectic structure. Since our functional integral has a four dimensional nature it turns out that the Liouville measure DA is the volume form associated to three different symplectic forms: ω_K and ω_C (ω_C is complex, so that its real and imaginary parts provide two real symplectic forms):

$$\begin{aligned} \omega_K &= \int d^4x \, \mathrm{Tr}(\delta A_z \wedge \delta A_{\bar{z}} + \delta A_u \wedge \delta A_{\bar{u}}) \\ \omega_C &= \int d^4x \, \mathrm{Tr}(\delta A_z \wedge \delta A_{\bar{u}}) \,. \end{aligned} \tag{3.4}$$

In fact the space of four dimensional connections can be thought of as an infinite dimensional affine hyper-Kahler manifold. For technical reasons that will be evident in section 4 we need a reduction of the structure (gauge) group from four to two dimensions. This is not restrictive in the large-N limit since it is known by Eguchi-Kawai that the theory can be reduced by quenching or twisting even to zero dimension keeping the same large-N limit. After the partial quenching (twisting) mentioned before, two components of the original connection still transform as a connection under the two dimensional structure group while the other two components transform as a Higgs field. We notice that the identification of two components of the gauge connection with the Higgs field is not canonical, i.e. coordinate independent. However in the case of QCD_4 we have a preferred choice of coordinates that corresponds to a flat metric on $T^2 \times T^2$ or better to a conformally flat metric, since the classical theory is conformally invariant in four dimensions. Our symplectic space can be now identified (non-canonically, i.e. in a coordinate dependent way) with the cotangent bundle of two dimensional connections, whose coordinates are the pairs $(A = A_z dz + A_{\bar{z}} d\bar{z}, \Psi = \Psi_z dz + \Psi_{\bar{z}} d\bar{z})$. More precisely in the partially quenched theory we have the identifications:

$$\begin{aligned} \Psi_z &= -iD_u = -i(\partial_u + iA_u) = p_u + A_u^{ch} \\ \Psi_{\bar{z}} &= i\bar{D_u} = i(\partial_{\bar{u}} - iA_{\bar{u}}) = p_{\bar{u}} + A_{\bar{u}}^{ch} \end{aligned} \tag{3.5}$$

while in the partially twisted theory we have:

$$\begin{aligned} \Psi_z &= \Gamma_u + A_u \\ \Psi_{\bar{z}} &= \Gamma_{\bar{u}} + A_{\bar{u}} \end{aligned} \tag{3.6}$$

with the other two components of the gauge connection remaining the same in both cases. Let us notice that we have used the quenching prescription to identify covariant derivatives with the Higgs field in the quenched theory and the twisting prescription for the analogous identification in the twisted theory. On the cotangent bundle the gauge transformations act by hamiltonian vector fields with respect to each of the symplectic structures [4, 25].

The gauge orbits of the hyper-Kahler reduction are classified by the set of levels of the moment maps (the hamiltonians) for the action of the gauge group, one for each of the three symplectic structures:

$$
\begin{aligned}
F_A - i\Psi^2 &= \mu^0 \\
\bar{\partial_A}\psi &= n \\
\partial_A\bar{\psi} &= \bar{n}
\end{aligned}
\tag{3.7}
$$

with μ^0 hermitean and n with values in $sl(N)$. Therefore the partially quenched EK model admits a stratification by the following microcanonical ensemble:

$$
Z = \int DAD\Psi D\mu^0 DnD\bar{n} \exp(-S_{EK})\delta(F_A - i\Psi^2 - \mu^0)\delta(\bar{\partial_A}\psi - n)\delta(\partial_A\bar{\psi} - \bar{n}) \,, \tag{3.8}
$$

where the obvious resolution of identity has been inserted. This equation has a meaning for every finite N in the quenched EK model, since the identification in eq. (3.5) has a meaning at the level of finite matrices. For the twisted model the preceding equations have to be modified because of the non-trivial commutator:

$$
[\Gamma_u, \Gamma_{\bar{u}}] = \theta^{-1}1 \tag{3.9}
$$

that requires in addition that the gauge orbits be infinite dimensional even before taking the large-N limit. In this case our microcanonical ensemble is given by:

$$
\begin{aligned}
F_A - i\Psi^2 &= \mu^0 + \theta^{-1}1 \\
\bar{\partial_A}\psi &= n \\
\partial_A\bar{\psi} &= \bar{n}
\end{aligned}
\tag{3.10}
$$

modulo gauge transformations, so that a central extension has been introduced in the hermitean moment map. From eq. (3.10) the microcanonical stratification for the partially twisted theory follows:

$$
Z = \int DAD\Psi D\mu^0 DnD\bar{n} \exp(-S_{TW})\delta(F_A - i\Psi^2 - \mu^0 - \theta^{-1}1)\delta(\bar{\partial_A}\psi - n)\delta(\partial_A\bar{\psi} - \bar{n}) \tag{3.11}
$$

The symplectic reduction in eq. (3.7) and (3.10) and the associated stratification of the functional integral are somehow formal, unless the structure of the moduli fields is specified. In general, if we require that the symplectic reduction be associated to a good moduli problem, i.e. separable, the moduli fields must be taken as a linear combination of delta functions, in order that the corresponding flatness condition defines a (possibly projective) representation of the fundamental group of the punctured torus (see appendix B). Despite this restriction, these moduli fields are dense in the sense of the distributions in the functional integral, that is all that we need to define it.

In section 4 we will see that the existence of punctures, i.e. of a boundary in our space-time torus, has a more fundamental meaning to solve loop equations than it seems from the definition of the microcanonical ensemble given here. We would like to terminate this

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section writing the resolution of identity associated to the microcanonical ensemble in a way that makes manifest its four dimensional euclidean (or Lorentz) covariant nature:

$$Z = \int DA_\mu D\mu^-_{\mu\nu} \exp(-S_{\rm YM})\delta(F^-_{\mu\nu}(A) - \mu^-_{\mu\nu} - (\theta^{-1})^-_{\mu\nu}1)\,, \tag{3.12}$$

where the $^-$ superscript refers to the projection into the anti-selfdual part of an anti-symmetric tensor. In eq. (3.12) we have written the resolution of identity of the twisted model, but the one of the quenched model can be obtained simply setting $(\theta^{-1})^-_{\mu\nu} = 0$ and interpreting covariant derivatives appropriately. In section 5 we will make use of this representation to get an explicit euclidean (or Lorentz) invariant form of the localisation integral.

4. Solving loop equations by Hitchin systems

In this section we consider loop equations for the connection $B = A + i\Psi$ in the partially quenched or twisted model. The advantage of using B as opposed to A_μ resides in the fact that planar loops for B generate a complete set of gauge invariant observables. The disvantage consists in the fact that B is not hermitean so that $\Psi(x, x; B)$ is not unitary in general. We will see later on under which conditions planar unitary monodromies, i.e. $\Psi(x, x; A)$, can be obtained directly from $\Psi(x, x; B)$. The loop equations for B read:

$$\begin{aligned} 0 = \int DB_\alpha \exp(-S_{\rm YM})\Bigg(&\mathrm{Tr}\left(\frac{\delta S_{\rm YM}}{\delta B_\alpha(z)}\Psi(x, x; B)\right) - \\ &- i\int_{C(x,x)} dy_\alpha \delta^{(2)}(z-y)\,\mathrm{Tr}(\Psi(x, y; B))\,\mathrm{Tr}(\Psi(y, x; B))\Bigg) \end{aligned} \tag{4.1}$$

with $\alpha = 1, 2$, so that in these variables they are as difficult to solve as for the original four dimensional connection A_μ. The quantum contribution in general is loop dependent and this dependence is an obstruction to the existence of a conventional (hyper-finite?) effective action (we take the point of view that an effective action defined directly in loop space [48, 49], from which the loop equations may be derived as critical equations, probably would not be hyper-finite).

After implementing in the functional integral the resolution of identity by means of the gauge orbits of the microcanonical ensemble of section 3, we change variables in the loop equations from Hitchin bundles to the corresponding holomorphic de Rham local system (see appendix B). In [59] an analogous change of variables, from the Hitchin system to the action-angle variables of the Hitchin fibration of the corresponding Dolbeault moduli space (see appendix B), was proposed on the basis of energy/entropy considerations.

Our basic philosophy consists in changing variables in such a way that the quantum contribution may be computed explicitly by evaluating a residue that depends only on the topological class of the loop. It is well known by the Cauchy formula that the line integral along a closed loop of a holomorphic function times the Cauchy kernel with pole not lying on the loop depends only on the winding number of the loop around the pole and on the value of the holomorphic function at the pole of the Cauchy kernel:

$$f(z)Ind_C(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{z-w}dw\,, \tag{4.2}$$

where $Ind_C(z)$ is the winding number of C around z. The existence would become possible of an effective action, for loops in a certain topological class determined by the winding number, if only the Cauchy kernel could be generated by functionally differentiating the monodromy of the connection in the loop equations with respect to the integration variables in the functional integral. We have seen in section 2 that functional differentiation with respect to the gauge connection produces a delta function in the quantum contribution to the loop equations. The change of variables to the de Rham systems is thought in such a way that functional differentiation of the monodromy with respect to the moduli fields of the holomorphic de Rham local systems produces precisely the Cauchy kernel in the loop equations. However, even if we reduce the computation of the quantum contribution to the evaluation of a line integral of a Cauchy kernel, there are two obstructions for the residue theorem to apply. The first one is that the monodromy is not a holomorphic function of its endpoints. The second one is that gauge invariance requires that the functional derivative with respect to the integration variable be taken at a point of the loop, for the result to be non-vanishing, i.e at a singularity point of the Cauchy kernel. We will see below how each of these obstructions is the cure for the other one. After introducing the non-hermitean connection $B = A + i\Psi = b + \bar{b}$ the microcanonical representation of section 3 holds in the form:

$$Z = \int \delta(F_B - \mu)\delta(d_A^*\Psi - \nu)\exp(-S_{EK})DBD\mu D\nu\,, \tag{4.3}$$

where we have re-casted the microcanonical resolution of identity in a different, equivalent form (see appendix B), for later convenience, in order to compute the effective action (section 5). We now change variables from the Hitchin bundles to the holomorphic de Rham local systems (see appendix B), whose moduli are described by the field μ', obtained from the equation:

$$F_B - \mu = 0 \tag{4.4}$$

by means of a complexified gauge transformation $G(x;B)$ that puts the connection $B = b + \bar{b}$ in the holomorphic gauge $\bar{b} = 0$:

$$\bar{\partial} b - i\mu' = 0\,, \tag{4.5}$$

where $\mu = \mu^0 + (n - \bar{n})$ and $\mu' = G\mu G^{-1}$. The partition function in terms of the moduli fields of the de Rham local system becomes:

$$Z = \int \delta(F_B - \mu)\delta(d_A^*\Psi - \nu)\exp(-S_{EK})DB\frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')}D\mu' D\bar{\mu}'\,. \tag{4.6}$$

The integral over B can now be performed and the resulting functional determinants, together with the jacobian of the change of variables to the de Rham local system, absorbed into the definition of the effective action Γ. Γ will be written more explicitly in the next section in terms of these functional determinants. Let us notice that for computing functional determinants a choice of metric is necessary. The natural metric for us is the translational invariant metric on the torus $\exp(-\rho)dzd\bar{z}$ with the conformal factor constant. However the classical action, being conformally invariant, does not depend on the choice of ρ. Yet we will see below that our regularization of the quantum contribution to the loop equations will require to transform the metric by a conformal diffeomorphism. The partition function is now:

$$Z = \int \exp(-\Gamma) D\mu' D\bar{\mu}' \,. \tag{4.7}$$

To realize our aim of getting the Cauchy kernel by functional differentiation with respect to the moduli fields of the de Rham local system it is convenient to study the loop equations for the Wilson loop constructed by the connection b, thought as a functional of B corresponding to gauge transforming B into the gauge $\bar{b} = 0$. Such a gauge transformation belongs to the complexification of the gauge group and it is rather a change of variables than a proper gauge transformation. However, because of the property of the trace, for closed loops, it preserves the trace of the monodromy. This will allow us, later in the course of our computations, to transform the loop equations thus obtained into equations for the monodromy of B. In our derivation of the loop equations, a crucial role is played by the condition that the expectation value of an open loop vanishes. For open loops constructed by means of the connection B this property follows directly from the local gauge invariance of the partially quenched or twisted EK model. However, the starting point in our case is the monodromy of the connection b, whose expectation value is also required to vanish for open loops. Yet, this condition does not follow from gauge invariance, since the change of variables to the de Rham local system ($\bar{b} = 0$) does not allow any residual local compact gauge freedom (unless perhaps we analytically continue to Minkowski space-time, see below). We will present here two slightly different derivations of the effective action each based on a different way of achieving the vanishing of the expectation value of open b loops.

The most direct procedure consists in observing that in the partially quenched theory the vanishing of the expectation value of open b loops follows from the fact that the partial quenching is thought precisely to keep unbroken the R^2 symmetry $B \to B + i(p_u dz + p_{\bar{u}} d\bar{z})$ and that in the gauge $\bar{b} = 0$ the R^2 symmetry is broken down to $b \to b + ip_u dz$. If this residual symmetry is unbroken, then the expectation value of the trace of open b loops vanishes because of the equation:

$$N^{-1} \langle \mathrm{Tr}\, \Psi(z_1, z_2; b) \rangle = \exp(p_u(z_1 - z_2)_z) N^{-1} \langle \mathrm{Tr}\, \Psi(z_1, z_2; b) \rangle \tag{4.8}$$

provided the normalised trace stays uniformly bounded in the large-N limit. To be precise, this holds only for loops that are open once projected on the z direction, because the R^2 symmetry is partially broken to $b \to b + ip_u dz$. However this weaker property is all that is needed in our loop equations. In the other proof of the loop equations that we propose below, the assumption of uniform boundness is not necessary and the vanishing of the expectation values of open b loops holds without further specifications. We may thus derive our loop equations in the partially quenched theory:

$$
\begin{aligned}
0 &= \int D\mu' D\bar{\mu}' \operatorname{Tr} \frac{\delta}{\delta\mu'(w)} (\exp(-\Gamma)\Psi(x,x;b)) \\
&= \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)} \Psi(x,x;b)\right) + \\
&\qquad\qquad + \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y) \operatorname{Tr}(\lambda^a \Psi(x,y;b)\lambda^a \Psi(y,x;b)) \Bigg) \\
&= \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)} \Psi(x,x;b)\right) + \\
&\qquad\qquad + \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y) \Bigg(\operatorname{Tr}(\Psi(x,y;b)) \operatorname{Tr}(\Psi(y,x;b)) - \\
&\qquad\qquad\qquad - \frac{1}{N} \operatorname{Tr}(\Psi(x,y;b)\Psi(y,x;b)) \Bigg) \Bigg)
\end{aligned} \tag{4.9}
$$

that in the large-N limit reduce to:

$$
\begin{aligned}
0 = \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)} \Psi(x,x;b)\right) + \\
+ \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y) \operatorname{Tr}(\Psi(x,y;b)) \operatorname{Tr}(\Psi(y,x;b)) \Bigg) .
\end{aligned} \tag{4.10}
$$

The most interesting case is when w lies on the loop C, because the loop equations can be transformed easily into an equation for B. However in this case the contour integration includes the pole of the Cauchy kernel. We need therefore a gauge invariant regularization. Before doing so it is convenient to make a conformal diffeomorphism to put the contour integral in standard form. We wish to do that because, when we regularize the theory, we require a loop-independent universal regularization. Hence it is natural to use the uniformization theory of Riemann surfaces with a boundary. It is not restrictive to take w as the base point of the loop. For every simple loop (i.e. non-overlapping, self-intersecting only at the base point) we make a conformal transformation that maps each petal into a disk and then each disk into the upper half-plane. Each petal of the loop is then mapped into a copy of the upper half plane and the boundary identified with the x-axis. Let t be the local coordinate on the upper half-plane corresponding to this conformal transformation. Obviously t depends on C and the microcanonical resolution of the identity in the functional integral, once written in term of t, depends on C too. The loop equations in the variables associated to the coordinate t look like the ones in the original global coordinate on the torus z. Therefore we get:

$$
\begin{aligned}
0 = \int D\mu'_{t\bar{t}} D\bar{\mu}'_{t\bar{t}} \exp(-\Gamma) \Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'_{t\bar{t}}(w)} \Psi(x,x;b)\right) + \\
+ \int_{C(x,x)} dy_t \bar{\partial}^{-1}(w-y) \operatorname{Tr}(\Psi(x,y;b)) \operatorname{Tr}(\Psi(y,x;b)) \Bigg)
\end{aligned} \tag{4.11}
$$

with the important difference that the contour integral is now restricted to the real axis and that the metric needed to define the functional determinants in Γ is the conformally transformed metric $\exp(-\rho)dtd\bar{t} = \exp(-\rho)|\frac{\partial t}{\partial z}|^2 dzd\bar{z}$. In $d = 4$ it is expected that this conformal transformation will affect the effective action only through its dependence on the conformal anomaly [60]. A natural regularization of the quantum contribution to the loop equations is a $i\epsilon$ regularization of the Cauchy kernel. This is the most natural among any other possible regularization in euclidean space, since it is in fact compatible with the analytic continuation to Minkowski space-time, that manifestly preserves gauge invariance. The idea that Feynmann diagrams in two dimensions could be regularized by analytical continuation to Minkowski space-time in certain cases is contained in [57]. A master field living in the Cuntz algebra in Minkowski space-time has been constructed in general, and in particular for Yang-Mills theories, in [61]. The result of the $i\epsilon$ regularization is the sum of two distributions, the principal part plus a one dimensional delta function:

$$\bar{\partial}^{-1}(w_x - y_x + i\epsilon) = (2\pi)^{-1}(P(w_x - y_x)^{-1} - i\pi\delta(w_x - y_x)) \,. \tag{4.12}$$

The loop equations thus regularized look like:

$$\begin{aligned} 0 = \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(&\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x,x;b)\right) + \\ &+ \int_{C(x,x)} dy_x (2\pi)^{-1}(P(w_x - y_x)^{-1} - i\pi\delta(w_x - y_x)) \times \\ &\times \operatorname{Tr}(\Psi(x,y;b)) \operatorname{Tr}(\Psi(y,x;b)) \Bigg) \,, \end{aligned} \tag{4.13}$$

where we are omitting the t under-script. Being supported on open loops the principal part does not contribute and the loop equations reduce to:

$$\begin{aligned} 0 = \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(&\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x,x;b)\right) - \\ &- \int_{C(x,x)} dy_x \frac{i}{2}\delta(w_x - y_x) \operatorname{Tr}(\Psi(x,y;b)) \operatorname{Tr}(\Psi(y,x;b)) \Bigg) \,. \end{aligned} \tag{4.14}$$

Taking $w = x$ and using the gauge transformation properties of the b monodromy and of $\mu(x)'$, the preceding equation can be rewritten in terms of the connection B and the moduli fields μ:

$$\begin{aligned} 0 = \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(&\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu(x)}\Psi(x,x;B)\right) - \\ &- \int_{C(x,x)} dy_x \frac{i}{2}\delta(x_x - y_x) \operatorname{Tr}(\Psi(x,y;B)) \operatorname{Tr}(\Psi(y,x;B)) \Bigg) \,, \end{aligned} \tag{4.15}$$

where we have used the condition that the trace of open loops vanishes to substitute the b monodromy with the B monodromy. This is our final form of the regularized euclidean loop equations. Let us notice that there is a sign ambiguity in the quantum contribution that depends on the choice of the sign of ϵ in the regulator or as to whether the conformal transformation is orientation preserving or orientation inverting. The occurrence of the factor of 1/2 seems to be related to the fact that the pole of the Cauchy kernel lies along the contour integral. If the contour integral were to include the pole inside the loop and the residue theorem to apply naively, the residue would be 1, while it would be 0 if the pole were outside the loop. Both choices however would violate gauge invariance.

For general self-intersection the euclidean loop equations reduce to (assuming the conformal map to be orientation preserving):

$$0 = \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(\operatorname{Tr} \left(\frac{\delta\Gamma}{\delta\mu(x)} \Psi(x,x;B) \right) - \\ - \frac{i}{2} \sum_{(y;y=x)} \operatorname{Ind}_{C_{(x,y)}}(y) \operatorname{Tr}(\Psi(x,y;B)) \operatorname{Tr}(\Psi(y,x;B)) \Bigg) . \quad (4.16)$$

Because of the product of traces, in general the quantum contribution does not have the same operator structure as does the classical term and it is loop dependent. Therefore, in general, an effective action need not to exist as a functional over the microcanonical ensemble constructed by the inductive limit of finite dimensional orbits. However, for self-avoiding loops, one of the traces in the quantum contribution is just $\operatorname{Tr}(1) = N$ and there is only one term as for the classical contribution. It is now clear the role played by the absence of self-intersection for these variables associated to the de Rham system of the microcanonical ensemble. It is only in this case that the quantum contribution is proportional to the monodromy as is the classical term. In general we get a loop dependent factorized trace of monodromies, that cannot be re-absorbed into a redefinition of Γ.

The result is the following euclidean master equation for self-avoiding loops:

$$0 = \int D\mu' D\bar{\mu}' \exp(-\Gamma) \left(\operatorname{Tr} \left(\left(\frac{\delta\Gamma}{\delta\mu(x)} - \frac{i}{2} \operatorname{Ind}_C(x) \operatorname{Tr}(1) \right) \Psi(x,x;B) \right) \right) \quad (4.17)$$

that reduces to:

$$0 = \frac{\delta\Gamma}{\delta\mu(x)} - \frac{i}{2} \operatorname{Ind}_C(x) \operatorname{Tr}(1) . \quad (4.18)$$

The corresponding effective action is:

$$\Gamma_{\text{eff}} = \Gamma - \epsilon(C) \frac{i}{2} \int \operatorname{Tr} \mu d^2x \quad (4.19)$$

with $\epsilon(C) = +1$ or -1. The two possible signs are due to the two possible orientations of the loop C. In addition there is a sign ambiguity due to the conventional choices that we made: the choice of regularization, i.e. $i\epsilon$ versus $-i\epsilon$ and the convention that the conformal map is orientation preserving. Remarkably the sign ambiguity in the euclidean loop equations corresponds in fact to physical equivalence. Indeed the quantum contribution is a θ-term with $\theta = \frac{1}{2}$. Now, since $\int \operatorname{Tr} \mu d^2x$ represents the first Chern class, θ is defined modulo integers, so that for $\theta = 1/2$ a change of sign does not change θ modulo 1. Had we regularized by analytical continuation to Minkowski space-time we would have obtained the same imaginary quantum contribution to the effective action, while the action Γ would have been Wick rotated by a factor of i. In any case we see that the quantum contribution for the de Rham system has produced a central extension in the master equation. This means in general, that even if we start with a microcanonical ensemble associated to $sl(N)$ bundles, central directions in the moduli fields have to be included, otherwise the master equation cannot be satisfied. Hence, in the partially quenched model we have to extend our microcanonical ensemble from $sl(N)$ to $gl(N)$ and to allow a central extension in the hermitean moment map as follows:

$$\begin{aligned} F_A - i\Psi^2 &= \mu^0 + c1 \\ \bar{\partial}_A \psi &= n \\ \partial_A \bar{\psi} &= \bar{n}\,. \end{aligned} \tag{4.20}$$

After that the entire chain of derivations of this section can be consistently performed to arrive at the master equation. Yet, the occurrence of the θ-term in the effective action is not completely harmless, since it affects the topological sectors of the U(N) theory (see for example [62]). In fact, we will see in section 5 that the central term in Γ_{eff} is cancelled by a counter-term, that arises as a $u(1)$-anomaly in the jacobian of the "chiral" change of variables from the Hitchin to the holomorphic de Rham systems, after a finite conformal rescaling of the metric used to compute the functional determinants. The "holographic" nature of our change of variables will be then completely manifest, in the sense that we are cancelling the quantum boundary contribution against one from the bulk, to obtain an effective action defined purely on the bulk.

Summarising, given a sequence of solutions of the master equation for any finite N, we have a sequence of solutions of the loop equations for finite N, up to terms vanishing in the large-N limit, corresponding to the omitted connected contribution. Therefore for N large but finite we have an approximation of the solution of the $N = \infty$ bare loop equations for planar self-avoiding loops with accuracy N^{-1}. The algebra generated by the monodromy of the master connection of the partially quenched EK model for any finite N is a matrix algebra, since the microcanonical ensemble defines a finite dimensional representation (of a central extension) of the fundamental group of a punctured torus. Obviously renormalization is necessary, since we have written the master equation for the bare master field, but in principle nothing forbids to perform it. Assuming convergence of the renormalised master monodromy as N goes to infinity, it follows the hyper-finiteness of the master algebra for planar closed self-avoiding B loops.

It is also clear that the twisted model leads to the same master equation, with the only difference that now the central extension is present from start and it is associated to an infinite dimensional projective representation of the fundamental group.
To show hyper-finiteness also in the twisted case we must associate to this infinite dimensional representation a system of approximating finite dimensional representations. In the twisted case it is not restrictive to start with a microcanonical ensemble that, though is

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infinite dimensional because of the central extension due to the non-trivial commutator:

$$[\Gamma_u, \Gamma_{\bar{u}}] = \theta^{-1} 1 \tag{4.21}$$

has finite rank residues at the punctures. This implies that the monodromies at the punctures are a direct sum of finite dimensional matrices and a multiple of identity. The representation obtained projecting away the multiple of identity part is the desired approximating system (finite dimensional approximations of infinite dimensional representations of the fundamental group have been previously considered in [63] in the framework of non-commutative Riemann surfaces and in the context of [59]). To be explicit the master algebra is the algebra generated by an inductive sequence of finite dimensional group algebras of representations of the fundamental group, where the inductive embedding is with respect to a partial order on the growing divisor, D, and on the increasing rank, N, of the local system. Thus we get in the quenched case:

$$\Psi(a)\Psi(b)\Psi(a)^{-1}\Psi(b)^{-1}\prod_p \Psi\left(D_p\right) = 1\,, \tag{4.22}$$

where the product is over the monodromies around the punctures and a and b are the cycles of the torus T^2. We assume translational invariance, that means that the monodromy around the punctures are constant up to a possible conjugacy transformation by means of the operator of translations of the quenched theory. In the twisted case the representation is projective and the constant monodromies may be conjugated either by the translation operator or by another twist operator:

$$\Psi(a)\Psi(b)\Psi(a)^{-1}\Psi(b)^{-1}\prod_p \Psi(D_p) = \exp\left(icV^{(2)}\right) 1\,, \tag{4.23}$$

where $V^{(2)}$ is the area of our space-time torus. The next version of the derivation of the effective action from the loop equations is based on the fact that for $\mathrm{U}(N)$ bundles, that we have in any case to introduce because of the occurrence of the central extension in the master equation, we can realize explicitly the R^2 symmetry as a global symmetry, even for the de Rham systems. In fact what really matters to generate the Cauchy kernel in the loop equations is the relation:

$$\bar{\partial} b - i\mu' = 0 \tag{4.24}$$

that follows from eq. (3.7) in the gauge $\bar{b} = 0$. Eq. (4.24) holds under slightly weaker conditions than $\bar{b} = 0$, that allow us to keep explicitly a residual R^2 global symmetry. For the Lie algebra $u(N)$ we decompose the connections B and b into their semi-simple and central parts: $B = B^s + B^c, b = b^s + b^c$. Our gauge fixing is now:

$$\begin{aligned} \partial \bar{b}^c &= 0 \\ \bar{b}^s &= 0\,. \end{aligned} \tag{4.25}$$

This gauge fixing admits the residual global symmetry $b \to b + ip_z dz, \bar{b} \to \bar{b} - ip_{\bar{z}} d\bar{z}$ maintaining at the same time the validity of eq. (4.24). If this symmetry is unbroken, then the expectation value of the trace of open b loops vanishes because of the equation:

$$\langle \Psi(x,y;b^c)\rangle = \exp(ip(x-y))\langle \Psi(x,y;b^c)\rangle\,. \tag{4.26}$$

We get:

$$0 = \int d^2p D\mu' D\bar{\mu}' \,\mathrm{Tr}\,\frac{\delta}{\delta\mu'(w)}(\exp(-\Gamma)\Psi(x,x;b))\,, \tag{4.27}$$

where we have inserted the integration over the zero mode d^2p. In the same vein as before we obtain:

$$\begin{aligned}
0 &= \int d^2p D\mu' D\bar{\mu}' \exp(-\Gamma)\Bigg(\mathrm{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x,x;b)\right) + \\
&\qquad\qquad + \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y)\,\mathrm{Tr}(\lambda^a\Psi(x,y;b)\lambda^a\Psi(y,x;b))\Bigg) \\
&= \int d^2p D\mu' D\bar{\mu}' \exp(-\Gamma)\Bigg(\mathrm{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(w)}\Psi(x,x;b)\right) + \\
&\qquad\qquad + \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y)\,\mathrm{Tr}(\Psi(x,y;b))\,\mathrm{Tr}(\Psi(y,x;b))\Bigg)\,.
\end{aligned} \tag{4.28}$$

Then the master equation follows as before. Several observations are in order.

An important feature of our regularized master equation is that it depends implicitly on the conformal diffeomorphism that maps the interior of the loop into the upper half plane. This is a direct consequence of our choice of regularising the quantum contribution to loop equations in a universal way. Indeed the local field for which the loop equations can be reduced to the master equation is the field $\mu_{t\bar{t}}$ that transforms as a $(1,1)$ form under conformal diffeomorphism. Instead $\mu_{z\bar{z}}$ is the field that occurs in the classical action, where $(z,\bar{z})$ are the global coordinates on the torus. The existence of this conformal mapping plays a key role when we introduce a lattice divisor, D, to get an explicit realization of our Hitchin system, actually the only one possible if we require a Hausdorff moduli space. In this case μ is a collective field given, according to section 3 and appendix B, by:

$$\begin{aligned}
\mu^0 &= \sum_p \mu^0_p \delta^{(2)}(x-x_p) \\
n &= \sum_p n_p \delta^{(2)}(x-x_p) \\
\bar{n} &= \sum_p \bar{n}_p \delta^{(2)}(x-x_p)\,.
\end{aligned} \tag{4.29}$$

Thus μ_p is a “lattice (or adelic) field of residues”. As a consequence we need a more sophisticated version of the uniformization map. In this case we ought to map conformally the interior of the loop minus the internal punctures of the lattice divisor into a punctured disk or into a fundamental region in the upper half-plane. In this case the image of the punctures in the upper half plane are cusps touching the real axis. The loop equations in terms of the "lattice field of residues" are thus:

$$0 = \int \prod_q D\mu_q' D\bar{\mu}_q{}' \exp(-\Gamma) \left(\mathrm{Tr}\left(\frac{\delta \Gamma}{\delta \mu_p} \Psi(x_p, x_p; B) \right) + \right. \tag{4.30}$$

$$\left. + \int_{C(x_p,x_p)} dy_z \bar{\partial}^{-1}(x_p - y)\, \mathrm{Tr}(\Psi(x_p, y; B))\, \mathrm{Tr}(\Psi(y, x_p; B)) \right) .$$

The cusp singularity at the point x_p introduces one more sign ambiguity for the central term in the effective action. In the lattice field version of our loop equations, represented by eq. (4.30), we must take by necessity the base-point of the loop at a puncture, otherwise we get a trivial result. This may seem unnatural from the point of view of the uniformization map, but somehow surprisingly this is precisely the presentation that occurs in Penner [64] definition of the Teichmuller space of bordered surfaces, whose moduli involve an arbitrary choice of a point on the boundary, that is then projected to infinity to become a cusp on the boundary of the upper-half plane. In appendix C we will see how the existence of the conformal diffeomorphism involved in the derivation of our master equation for the field of residues seems to be crucial to get the correct area law for the Wilson loop in two dimensions by the Hitchin systems.

In a sense we may consider the effective action as the image of a "holographic" map from the boundary, i.e. the loops, to the bulk, i.e. the torus T^2, the space-time of the quenched theory. From this point of view the HF property of the master algebra is related to how much entropy is encoded into the boundary. For self-avoiding loops we have a HF effective action, while for arbitrary self-intersection of the boundary we may need a non-HF algebra.

The preceding derivation of the master equation from the loop equations literally allows us to employ the master field B, that is the solution of eq. (4.30), only to compute the corresponding Wilson loop $\lim_N N^{-1} \mathrm{Tr}\, \Psi(x, x; B)$. However the quantity that has most physical interest is rather $\lim_N N^{-1} \mathrm{Tr}\, \Psi(x, x; A)$, i.e. the trace of the unitary monodromy. This quantity can be computed via a limiting procedure taking advantage of the existence of the following κ action ($\psi \to \kappa\psi$) on the Hitchin bundles [5] (see appendix B). After some obvious rescaling in the defining equations of our microcanonical ensemble we get a κ dependent effective action appropriate for the study of the monodromy of the operator $B^\kappa = A + i(\kappa\psi + \bar{\kappa}\bar{\psi}) = A^\kappa + i\Psi^\kappa$:

$$\begin{aligned} Z &= \int \delta\left(F_{B^\kappa} - \mu - \theta^{-1}|\kappa|^2 1\right) \delta(d^*_{A^\kappa}\Psi^\kappa - \nu) \exp(-S_{TW}) DB \frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')} D\mu' D\bar{\mu}' \\ &= \int \exp(-\Gamma^\kappa) D\mu' D\bar{\mu}' . \end{aligned} \tag{4.31}$$

After that the monodromy of A is recovered in the limit $\kappa \to 0$. This poses the interesting problem as to whether there exist solutions for A and Ψ that are κ independent, as it would be required by the spirit if not the letter of the master field philosophy, and more generally euclidean (or Lorentz) covariant, in the sense that the euclidean or Lorentz group are embedded into the (global) gauge group [10]. As far as κ-independence is concerned it is clear that any κ-independent solution must coincide with a fixed point of the κ action as $\kappa \to 0$ on the infinite dimensional Hitchin bundles involved. The study of the limit $\kappa \to 0$ would also furnish the Wilson loop for the A monodromy. We are therefore interested to study the κ action on infinite dimensional Hitchin systems, a case on which we have not much information. However in finite dimensions such fixed points exist and form a lagrangian subspace of the moduli of Hitchin bundles that coincides with those bundles that arise as variations of Hodge structures (see appendix B) [65]–[67]. For such bundles the B monodromy coincides with the A monodromy [65]–[68] (see appendix B). As far as euclidean or Lorentz covariance is concerned, bundles that arise as variations of Hodge structures are associated to embeddings of $sl(2,R)$ into $sl(N,C)$ via the Schmid nilpotent orbit theorem [65]–[67]. It is natural to speculate, taking into account also the peculiar interplay between the euclidean and Minkowski version of the master equation, that such embeddings may be related to embeddings of the Lorentz group into the gauge group. In addition we observe that a necessary condition for euclidean or Lorentz covariance is given by the vanishing in the large-N limit of the central extension in the microcanonical ensemble for the master field. Indeed any central element can only transform as a scalar under euclidean or Lorentz transformations, if these are to be embedded into the gauge group, while this contradicts the nature of the defining equations of the microcanonical ensemble, that transform as a anti-selfdual tensor in our four dimensional interpretation. We have already seen that the effective action develops a central term as result of the quantum contribution. However this term must be an artifact due to the "chirality" of our change of variables. Therefore it must be cancelled by the contribution of an "anomalous" jacobian of the change of variables in the bulk. In the next section we will see that Γ contains a counter-term that cancels the quantum contribution provided a suitable finite rescaling of the conformal factor in the metric is made (indeed the "anomalous" jacobian depends on the conformal factor of the metric, once the complex structure is fixed, while the quantum contribution to the effective action, being a residue, does not.)

5. The effective action of large-N QCD_4 and of its $\mathcal{N} = 1$ extension

We are now ready to look at the effective action for Hitchin systems in some detail. It easy to see that the effective action, as it would be computed from the loop equations without gauge fixing, is the sum of the classical (quenched or twisted) Yang-Mills action plus the logarithm of a number of determinants whose origin is the following one.

First of all there are the determinants that arise from the localisation integral on the microcanonical ensemble, i.e. from integrating the delta functionals in eq. (3.8) and (3.11) over the space of connections:

$$\mathrm{Det}'(d_B \wedge)^{-1}\, \mathrm{Det}'(\mathrm{Re}\, d_B^*)^{-1}\, \mathrm{Det}'(p^*\omega)\,. \tag{5.1}$$

The first two determinants arise from integrating over the tangent space to the Hitchin equations. The third one is the pull back, on the moduli that are not local moduli, of the symplectic form on the space of connections. According to the formulas on the dimension of the moduli space in appendix B, there are not such non-local moduli on a torus or on a sphere, once the local moduli fields are fixed, while there are on Riemann surfaces of higher genus. Therefore for the quenched or twisted theory on a torus the third factor is absent. The $'$ suffix refers to the need of projecting away from the determinants the zero modes due to gauge invariance, since gauge-fixing is not implied in the loop equations, though it may be understood if we like to. Finally there is the determinant that arises as the jacobian of the change of variables from the local moduli fields of the Hitchin system to the corresponding holomorphic de Rham local system:

$$\mathrm{Det}' \, \frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')} \,. \tag{5.2}$$

At formal level and without an explicit gauge fixing, the effective action, Γ, reads:

$$\Gamma = S_{\mathrm{YM}} - \log \mathrm{Det}'(d_B \wedge) - \log \mathrm{Det}'(\mathrm{Re}\, d_B^*) + \log \mathrm{Det}' \, \frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')} \,, \tag{5.3}$$

where S_{YM} is the Yang-Mills action of the reduced quenched or twisted theory. The effective action can be given a non-formal meaning introducing a metric, to make sense of the formal determinants, and making a choice of gauge by the Faddeev-Popov procedure, to avoid the zero modes. A convenient choice of gauge is [69]:

$$d_A^* \delta A + i a d_\Psi^* \delta \Psi = 0 \tag{5.4}$$

that coincides with a background euclidean or Lorentz covariant gauge (after analytical continuation to Minkowski space-time). This gauge fixing condition can be written in terms of B as [69]:

$$\mathrm{Re}(d_B^* \delta B) = 0 \,. \tag{5.5}$$

After introducing this gauge-fixing and the corresponding Faddeev-Popov determinant, the localisation integral over the microcanonical ensemble gives us:

$$\mathrm{Det}(-\Delta_1(B))^{-1/2}\, \mathrm{Det}(-\mathrm{Re}\,\Delta_B)^{-1/2}\, \mathrm{Det}(-\mathrm{Im}\,\Delta_B)^{-1/2}\, \mathrm{Det}(-\mathrm{Re}\,\Delta_B)^{-1} \,, \tag{5.6}$$

where Δ_B is the scalar laplacian and $\Delta_1(B)$ is the operator:

$$\Delta_1(B) = d_B^* d_B \wedge \tag{5.7}$$

acting on one-forms and restricted to $d_B^* \delta B = 0$. The operator $-\Delta_1(B)$ arises from interpreting $\int DB \delta(F_B - \mu)$ as:

$$\lim_{\epsilon \to 0} \int DB \exp\left(-\frac{1}{\epsilon}\, \mathrm{Tr}((F_B - \mu)(F_B - \mu)^*)\right) \tag{5.8}$$

with the constraint $d_B^* \delta B = 0$. The last determinant in eq. (5.6) is the Faddeev-Popov determinant. The last three factors in eq. (5.6) cancel together. The effective action is thus:

$$\Gamma = S_{\mathrm{YM}} - \frac{1}{2} \log \mathrm{Det}(-\Delta_1(B)) + \log \mathrm{Det}\, \frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')} \,. \tag{5.9}$$

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The first determinant, that we obtained from the microcanonical localisation, is essentially a one-loop contribution in a background field. We should notice that it contributes two hermitean polarisations. In fact the connection B is non-hermitean and thus it is determined by four real polarisations, but two of them are killed by the conditions:

$$\begin{aligned} \mathrm{Re}(d_B^* \delta B) &= 0 \\ \mathrm{Im}(d_B^* \delta B) &= 0 \end{aligned} \tag{5.10}$$

that come from the gauge-fixing and from the harmonic constraint [69] in the functional integral. However this determinant can be more conveniently defined and computed by means of a manifestly euclidean (or Lorentz) invariant metric and of the representation of the microcanonical ensemble reported at the end of section 3. Though we use now a manifestly euclidean (or Lorentz) invariant notation, there is no problem to interpret the following formulas appropriately for the quenched or for the twisted models. The localisation integral in eq. (3.12) is defined formally as:

$$\int DA_\mu \delta(F^-_{\mu\nu} - \mu^-_{\mu\nu}) = \mathrm{Det}'^{-1}(P^- d_A \wedge) = \mathrm{Det}'^{-1/2}(-\Delta_1)\,, \tag{5.11}$$

where P^- is the projector onto the anti-selfdual part of the field strength. After inserting the gauge-fixing condition and the corresponding Faddeev-Popov determinant it can be defined non-formally as:

$$\begin{aligned} \mathrm{Det}^{-1/2}(-\Delta_1) = &\lim_{\epsilon\to 0} \int DcDA_\mu \exp\left(-\frac{1}{\epsilon}\,\mathrm{Tr}(c^2)\right) \times \\ &\times \exp\left(-\frac{1}{\epsilon}\,\mathrm{Tr}((F^-_{\mu\nu} - \mu^-_{\mu\nu})^2)\right) \delta(D_\mu \delta A_\mu - c)\Delta_{FP}\,. \end{aligned} \tag{5.12}$$

The result for $\mathrm{Det}^{-1/2}(-\Delta_1)$ is thus:

$$\mathrm{Det}^{-1/2}(-\Delta_1) = \mathrm{Det}^{-1/2}(-\Delta_A \delta_{\mu\nu} 1 + i\,ad_{F^-_{\mu\nu}})\Delta_{FP}\,, \tag{5.13}$$

where $F^-_{\mu\nu}$ is the anti-selfdual part of the field strength, given by:

$$F^-_{\mu\nu} = F_{\mu\nu} - F^*_{\mu\nu} \tag{5.14}$$

with:

$$F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}\,. \tag{5.15}$$

It is a fundamental problem to study the renormalization group properties of the master equation for our effective action, with special regard to the asymptotic freedom [70, 71]. In appendix D we have computed the divergent part of the localisation determinant $\mathrm{Det}^{-1/2}(-\Delta_1)$ considered as a functional of the anti-selfdual $F^-_{\mu\nu}$ or self-dual field strength $F^+_{\mu\nu}$:

$$\begin{aligned}
-\frac{1}{2}\log\mathrm{Det}(-\Delta_1) &= -\frac{11}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2 + \\
&\quad +\frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^+_{\mu\nu})^2 \\
&= -\frac{12}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2 + \\
&\quad +\frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2\,.
\end{aligned} \tag{5.16}$$

We should notice that the logarithmic divergence associated to $\mathrm{Tr}(F^+_{\mu\nu})^2$ really occurs in the effective action only if the effective action is regarded as a local functional of $F^-_{\mu\nu}$ and $F^+_{\mu\nu}$. Yet in the master equation we should rather consider the effective action as a functional of the moduli fields. Now the components of the tensor $F^-_{\mu\nu}$ are certainly a local functional of the moduli fields, since the moduli fields are a linear combination of them, while $F^+_{\mu\nu}$ is not, since:

$$P^-(d_A\delta A) = \delta\mu^-\,. \tag{5.17}$$

Because of this non-local character it is not obvious that, once the non-local expression in terms of the moduli fields is inserted in the definition of the localisation determinant, it produces a logarithmic divergence. In any case the master equation reads:

$$\frac{\delta\Gamma_{\text{eff}}}{\delta\mu(x)} = 0\,. \tag{5.18}$$

The left hand side contains, by the definition of the moduli fields μ, a linear combination of:

$$\frac{\delta\Gamma_{\text{eff}}}{\delta\mu^-_{\mu\nu}(x)}\,. \tag{5.19}$$

Using eq. (5.17) we get $\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2 = \int d^4x\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2$ at lowest order in powers of $\mu^-_{\mu\nu}$, assuming euclidean (or Lorentz) invariance (of the quenched theory). It follows that, at first order in the expansion in powers of the moduli fields, which by the way coincides at this order with the usual perturbative expansion, the contribution of the self-dual part of the effective action vanishes. The first coefficient of the beta function is thus exactly reproduced at this order by the localisation determinant (for the moment we do not include the contribution of $\log\mathrm{Det}\,\frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')}$ whose computation we leave for the future):

$$\frac{\delta}{\delta\mu(x)}\int d^4x\left(\frac{N}{2\lambda}\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2 - \frac{11}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2\right) = 0\,. \tag{5.20}$$

The preceding construction of the effective action of QCD_4 extends easily to the supersymmetric case without scalars. In fact, if only fermions are present in addition to the gauge fields, they can be integrated out, the result being a pfaffian whose logarithm should be added to the effective action. Thus in the $\mathcal{N}=1$ case, we get for the effective action:

$$\Gamma = S_{\text{YM}} - \frac{1}{2}\log\mathrm{Det}(-\Delta_1(B)) + \log\mathrm{Det}\,\frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')} + \log Pf(\sigma_\mu D_\mu)\,. \tag{5.21}$$

We must now give an interpretation of the conformal diffeomorphism, due to the uniformization map, compatible with the four dimensional interpretation of the quenching or twisting procedure. Thus the dependence on the conformal factor in the first determinant of eq. (5.9) will be only through the conformal anomaly, since it is the determinant of a laplacian acting on one-forms in four dimensions (though the laplacian looks two dimensional, we have seen that the missing two dimensional integration is hidden into the definition of the trace and in the definition of Ψ in the twisted theory or into the dimensionful rescaling of the coupling constant and in the quenching prescription for Ψ in the quenched case). The second determinant in eq. (5.9) instead depends explicitly on the conformal factor of the metric since it is the determinant of an operator acting on scalars. One of the features that distinguishes the present approach from the microcanonical localisation, is the contribution from the jacobian of the change of variables from Hitchin bundles to the de Rham systems:

$$\mathrm{Det}\,\frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')} = \mathrm{Det}^{-1}\,\frac{D(\mu',\bar{\mu}')}{D(\mu,\nu)} = \mathrm{Det}^{-1}\begin{pmatrix} \frac{D\mu'}{D\mu} & \frac{D\mu'}{D\nu} \\ \frac{D\bar{\mu}'}{D\nu} & 0 \end{pmatrix}, \tag{5.22}$$

where to get the zero in eq. (5.22) we used the relations:

$$\begin{aligned} d_B^*\delta B &= i\delta\nu \\ d_B \wedge \delta B &= \delta\mu\,. \end{aligned} \tag{5.23}$$

This determinant is in fact an "anomalous" jacobian for the action of the two dimensional complexified gauge group. This is reminiscent of the fact that the analogous jacobian that occurs in the collective field method applied to loop equations arises as an anomaly in the algebra of loop substitutions [72]. More explicitly we can extract from $\log\mathrm{Det}\,\frac{D(\mu,\nu)}{D(\mu',\bar{\mu}')}$ a term linear in $\mathrm{Tr}\,\mu$ as follows:

$$\begin{aligned} \mathrm{Det}\left(\frac{D\mu}{D\mu'}\right) &= \mathrm{Det}^{-1}\left(\frac{D\mu'}{D\mu}\right) \\ &= \mathrm{Det}^{-1}\left(\frac{DG\mu G^{-1}}{D\mu}\right) = \mathrm{Det}^{-1}\left(1+\left[\mu, G^{-1}\frac{DG}{D\mu}\right]\right). \end{aligned} \tag{5.24}$$

We now use the formula:

$$\mathrm{Det}(1+A) = \exp\mathrm{Tr}\log(1+A)\,. \tag{5.25}$$

Since A in our case is a commutator, in finite dimension the contribution linear in A would vanish. However this is not anymore true in infinite dimensions, that is our case, since the trace includes integration over space-time. We will apply the general theory of regularized determinants to the case that A is a pseudo-differential operator. According to the general theory [73, 74], the definition of regularized determinant employs a generalised noncyclic trace $\mathrm{TR}_{\mathrm{reg}}$. The definition of $\mathrm{TR}_{\mathrm{reg}}$ involves a choice of a positive pseudo-differential operator Q of positive order q.

$$\mathrm{TR}_{\mathrm{reg}}(A) = \lim_{\epsilon\to 0}\mathrm{Tr}(\exp(-\epsilon Q)A) \tag{5.26}$$

or equivalently (the two definitions are related by a Mellin transform):

$$\mathrm{TR}_{\mathrm{reg}}(A) = \lim_{z\to 0} \mathrm{Tr}(Q^{-z}A) \tag{5.27}$$

$\mathrm{TR}_{\mathrm{reg}}$ is a non-cyclic trace, so that, in general, the trace of a commutator does not vanish. The renormalised trace is instead:

$$\mathrm{TR}_{\mathrm{ren}}(A) = \lim_{z\to 0} \mathrm{Tr}\left(Q^{-z}A - \frac{1}{zq}\mathrm{res}(A)\right), \tag{5.28}$$

where $\mathrm{res}(A)$ is the Wodzicki residue of a pseudo-differential operator. It is defined as:

$$\mathrm{res}(A) = \frac{1}{(2\pi)^d}\int_M dx \int_{|p|=1} \mathrm{tr}a_{-d}(x,p)\,, \tag{5.29}$$

where $a_{-d}(x,p)$ is the homogeneous term of order $-d$ in the momenta, in the asymptotic expansion of the matrix valued symbol of A. The Wodzicki residue corresponds to extract the coefficient of the logarithmically divergent term in the symbol expansion of a pseudo-differential operator. Since usually anomalies arise from finite integrals, for anomalies the regularized and renormalised traces coincide. An important formula to compute the regularized trace is:

$$\mathrm{TR}_{\mathrm{reg}}[A,B] = -\frac{1}{q}\mathrm{res}([\log|Q|,A]B)\,. \tag{5.30}$$

We are now ready to discuss the contribution of the chiral jacobian $\mathrm{Det}(\frac{D\mu}{D\mu'})$. We anticipated that in the effective action the central term that comes from the residue in the quantum contribution to loop equations must be cancelled by an analogous term from the chiral "anomalous" jacobian, since QCD is non-chiral and there is no central term in the defining classical theory. This anomalous term in the chiral jacobian, that is the trace of the term linear in μ:

$$\frac{1}{q}\mathrm{res}\left(\left[\log|Q|, G^{-1}\frac{DG}{D\mu}\right]\mu\right) \tag{5.31}$$

defines a cocycle, that coincides with the Radul cocycle. Now the commutator anomaly for point-wise multiplication vanishes in every dimension but one, where coincides with the (twisted) Radul cocycle. However our commutator involves an operator, $G^{-1}\frac{DG}{D\mu}$, whose symbol is not a local operator, since the functional derivative of the gauge transformation with respect to μ involves the inverse laplacian. Thus in our case the commutator anomaly does not vanish in dimension two, if we take as Q the two dimensional laplacian, a choice that seems appropriate in the quenched theory. Thus we may compute the $u(1)$ part of our chiral anomaly as follows (for the metric $dzd\bar{z}$):

$$\begin{aligned}\frac{1}{q}\mathrm{res}\left(\left[\log|Q|, G^{-1}\frac{DG}{D\mu}\right]\mu\right) &= \frac{i}{4}\mathrm{res}\left(\left[\log|\partial\bar{\partial}|, (\partial\bar{\partial})^{-1}\right]\mu\right)\\ &= \frac{i}{2\pi}\int d^2x\,\mathrm{Tr}(\mu(x))\,.\end{aligned} \tag{5.32}$$

From eq. (5.32) we see that there exists a finite mismatch of a factor of π between the commutator anomaly and the quantum central term in the effective action. However the commutator anomaly depends on the choice of the metric, in particular of the conformal factor of the metric, while the metric and a fortiori its conformal factor do not affect the residue in the quantum term. Therefore we are able to cancel the central term in the effective action in the quenched theory by a conformal rescaling of the metric of a factor of π. It would be interesting to perform the analogous computation in the twisted theory. In four dimensions a finite conformal rescaling of the metric changes the effective action only by finite terms due to the four dimensional conformal anomaly. The same happens in two dimensions through a different mechanism though: since the classical action depends on the conformal scale of the metric, a conformal rescaling is equivalent to a rescaling of the dimensionful coupling constant (see appendix C). To be precise the sign of the commutator anomaly in the euclidean space-time is fixed, while it is not so for the sign of the quantum contribution, so that the cancellation between them can actually occur only modulo integers, but in fact this is all that is needed, since the θ-term in the effective action is defined modulo integers too.

6. Loop equations and the geometric Langlands correspondence

We have seen in the previous section that loop equations admit as solutions inductive limits of (projective) representations of the fundamental group of a punctured torus. In this section we will use the geometric Langlands correspondence in the sense of Beilinson and Drinfeld [26, 27] (and references therein) to interpret the master equation by operator solutions of the loop equations obtained by the quantisation of the Hitchin systems. As we will see the construction that we are referring to is in a certain sense dual to the one of section 4. Because of quantisation, the HF of the ambient algebra of the master field is not preserved a priori, while it is kept quantum integrability. We have seen that the loop equations:

$$0 = \int D\mu' D\bar{\mu}' \exp(-\Gamma) \Bigg(\mathrm{Tr}\left(\frac{\delta\Gamma}{\delta\mu(w)} \Psi(x,x;B) \right) + \\ + \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y)\, \mathrm{Tr}(\Psi(x,y;B))\, \mathrm{Tr}(\Psi(y,x;B)) \Bigg) \tag{6.1}$$

for self-avoiding loops are implied by the master equation:

$$0 = \frac{\delta\Gamma_{\text{eff}}}{\delta\mu(x)} \,. \tag{6.2}$$

However there is a different way of satisfying the loop equations. We may think that the functional integration in the loop equations defines a generalised trace $\langle \Omega \mid\mid \Omega \rangle$ on the field algebra. Quantisation in presence of constraints, that in our case are Hitchin self-duality equations on a punctured torus, is usually performed in two ways. Either we first quantise the space of connections [35, 36] and then we impose the constraints on the physical states of the theory or we first solve the constraints and then we quantise the reduced theory. In our case we must follow the second alternative because of the localisation over the microcanonical ensemble in the functional integral. It is well known that if the moduli are quantised they are interpreted as differential operators in some representation of the universal enveloping algebra [35, 36, 76] and [28]–[32]. Thus a solution of the loop equations can be found provided we find a trace $\langle\Omega \mid\mid \Omega\rangle$ that satisfies:

$$\begin{aligned} 0 = \langle\Omega \mid & \frac{\delta\Gamma}{\delta\hat{\mu}(w)}\Psi(x,x;\hat{B}) \mid \Omega\rangle + \\ & + \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y)\langle\Omega \mid \Psi(x,y;\hat{B}) \mid \Omega\rangle\langle\Omega \mid \Psi(y,x;\hat{B}) \mid \Omega\rangle \end{aligned} \tag{6.3}$$

for self-avoiding loops, that certainly happens if:

$$0 = \langle\Omega \mid \frac{\delta\Gamma_{\text{eff}}}{\delta\hat{\mu}(x)}\,, \tag{6.4}$$

where we have introduced the $\hat{}$ superscript to indicate that moduli are now differential operators on a Hilbert space. This is our new version of the master equation. In principle we could parameterise the state $\langle\Omega \mid$, that we are looking for, by means of the spectrum of the hamiltonians $\hat{H}_i$ of the quantum Hitchin system (see appendix B), following the technique used in [28] to solve the Bethe ansatz equations for the quantum Hitchin hamiltonians (on a punctured sphere). The basic idea there is to interpret Hitchin hamiltonians as elements of the centre of a Kac-Moody algebra at critical level, in a way that we recall briefly below (the trace property should be checked separately, using perhaps the quasi-conformal structure associated to the critical level [31, 32]). Langlands duality is a "far reaching generalisation of Artin-Hasse theory in number field" [77] that loosely speaking associates to every adelic automorphic representation on a local field a representation of its Galois group. Beilinson and Drinfeld started a program of understanding Langlands duality from a geometrical point of view. From this point of view to the Galois side corresponds a representation of the fundamental group of a Riemann surface, while to the automorphic side correspond certain D-modules on the moduli stack of vector bundles on a Riemann surface. More precisely Beilinson and Drinfeld proved that the spectrum of the set of commuting operators obtained quantising the Hitchin hamiltonians (i.e. the D-modules on the automorphic side) is parameterised by opers with values in the dual Langlands group [26]. To the opers is then associated a representation of the fundamental group via the monodromy representation (this is the Galois side). The D-module associated to the quantisation of Hitchin bundles is in fact a representation of a Kac-Moody algebra of critical level, actually of its centre [28, 31, 32]. It is known that the common eigenstates of the set of Hitchin hamiltonians can be obtained by the Bethe ansatz [28] (see also [76]).

It is also clear that the first variation of the effective action computed at the master field can be considered as an element of the universal enveloping algebra of the global gauge group, because of translational invariance. But the universal enveloping algebra of any Lie group can be embedded into the enveloping algebra of the corresponding Kac-Moody algebra via the constant matrices. Therefore we can add to the Hitchin hamiltonians, i.e. to the centre of the universal enveloping algebra of the Kac-Moody algebra of critical level, just one operator in the universal enveloping algebra and still get a commutative

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family. We choose to add precisely the first variation of the effective action associated to the quantised Hitchin system. We can then solve the master equation:

$$0 = \langle \Omega \mid \frac{\delta \Gamma_{\text{eff}}}{\delta \hat{\mu}(x)} , \tag{6.5}$$

where $\langle \Omega \mid$ varies over the family of the common eigenstates of the quantum Hitchin hamiltonians that are parameterised by the moduli of opers for the dual Langlands group:

$$\langle \Omega \mid \lambda_i = \langle \Omega \mid \hat{H}_i \, . \tag{6.6}$$

Thus we come to the conclusion that while in section 4 the master field was parameterised by the moduli of the representations of the fundamental group with values in the global gauge group, here the vacuum is parameterised by the moduli of representations in the dual Langlands group. It is tempting to identify Langlands duality with 't Hooft electric-magnetic duality since the U(1) subgroups of the gauge group are mapped into each other by Langlands duality precisely in the same way abelian electric-magnetic duality does. We should mention that there is a natural physical interpretation of this construction in terms of the conjectured existence of the gauge-field/string correspondence [33, 34]. The main difference, apart from the one already mentioned about quantisation, is that if we insist to interpret Hitchin hamiltonians as elements of the centre at critical level, there is no associated conformal vertex algebra, but rather only a quasi-conformal one into which the master field would live inside [31, 32]. We should also mention that the construction of section 4 furnishes, after quantisation, two different representations of the master field, related by a non-unitary similarity transformation. Indeed the master field b in the holomorphic gauge would be an element of a chiral vertex algebra, while the master field B in the unitary gauge would be an element of a non-chiral one. While this does not affect the Wilson loop, because of the required trace property of $\langle \Omega \| \Omega \rangle$, some non-holomorphic information (i.e. the harmonic metric) is necessary to compute the effective action Γ_{eff}.

7. Conclusions

It is known that the von Neumann algebra generated by the master field of QCD in d dimensions, for d larger or equal to two, or by any equivalent Eguchi-Kawai reduction to any lower dimension, is a sub-algebra of a Cuntz algebra with more than one self-adjoint generator, that is not hyper-finite, i.e. it is not the weak limit of a sequence of finite dimensional matrix algebras. Yet functional techniques have shown that for QCD_2 some special observables, such as self-avoiding Wilson loops, admit a description in terms of a master field that generates a hyper-finite (abelian) von Neumann algebra. We have shown in this paper that for planar self-avoiding Wilson loops, constructed by the connection $B = (A_z + \kappa D_u)dz + (A_{\bar{z}} - \bar{\kappa}\bar{D}_u)d\bar{z}$ with $\kappa \neq 0$ in a two dimensional Eguchi-Kawai model equivalent to the large-N limit of QCD_4, the large-N loop equations admit as solution a master monodromy that, assuming existence of the large-N limit, generates a hyper-finite von Neumann algebra. An analogous construction holds for a partially twisted version, i.e. for the equivalent non-commutative gauge theory. In particular the master algebra of the partially quenched or twisted EK model is the limit group algebra of a sequence of (possibly projective) representations (with values in the complexified gauge group) of the fundamental group of an arbitrarily punctured torus. As such it can be related to infinite dimensional Hitchin systems and in fact the loop equations written in terms of the "adelic field of moduli" of the Hitchin systems:

$$0 = \int \prod_q D\mu'_q D\bar{\mu}_q{}' \exp(-\Gamma) \Bigg(\operatorname{Tr}\left(\frac{\delta\Gamma}{\delta\mu_p} \Psi(x_p, x_p; B) \right) + \tag{7.1}$$
$$+ \int_{C(x_p,x_p)} dy_z \bar{\partial}^{-1}(x_p - y) \operatorname{Tr}(\Psi(x_p, y; B)) \operatorname{Tr}(\Psi(y, x_p; B)) \Bigg)$$

are shown to be implied, for planar self-avoiding loops, by a master equation that determines the critical set of the following effective action:

$$\Gamma_{\text{eff}} = S_{\text{YM}} - \frac{1}{2} \log \operatorname{Det}(-\Delta_1(B)) + \log \operatorname{Det} \frac{D(\mu, \nu)}{D(\mu', \bar{\mu}')} - \epsilon(C) \frac{i}{2} \int \operatorname{Tr} \mu d^2x \,. \tag{7.2}$$

Remarkably the first determinant in eq. (7.2) can be computed in a manifestly four dimensional euclidean (or Lorentz) invariant way and, once inserted into the master equation, implies the correct value of the one-loop beta function up to terms of higher order in powers of the moduli fields.

There is a dual construction, in the sense of the geometric Langlands correspondence, in which the Hitchin systems are quantised, the moduli are differential operators and the effective action is an element of the enveloping algebra of the global gauge group. Thus the loop equations:

$$0 = \langle \Omega \mid \frac{\delta\Gamma}{\delta\hat{\mu}(w)} \Psi(x, x; \hat{B}) \mid \Omega \rangle +$$
$$+ \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w - y) \langle \Omega \mid \Psi(x, y; \hat{B}) \mid \Omega \rangle \langle \Omega \mid \Psi(y, x; \hat{B}) \mid \Omega \rangle \tag{7.3}$$

are transformed, for self-avoiding loops, into the condition that the master equation annihilates the vacuum:

$$0 = \langle \Omega \mid \frac{\delta\Gamma_{\text{eff}}}{\delta\hat{\mu}(x)} \,. \tag{7.4}$$

A possible physical interpretation of this quantised construction is in terms of a quasi-conformal (vertex) algebra relative to the conjectured string/gauge-field duality.

A. Von Neumann algebras

In this appendix we follow [41].

A von Neumann algebra is a sub-algebra of the algebra of all bounded operators on a Hilbert space that is closed under the adjoint operation (*) and under taking limits in the topology of weak convergence, the weak topology in short. The dual of a von Neumann algebra is the set of all bounded linear functionals over the algebra. It is usually assumed that the Hilbert space over which the algebra acts has a countable basis and that the von Neumann algebra has a separable dual. Von Neumann double commutant theorem asserts that a von Neumann algebra coincides with the commutant of its commutant. The commutant is the set of all bounded operators that commute with all the operators of the algebra. The centre of an algebra is the set of all the operators of the algebra that commute with all operators of the algebra.

A von Neumann algebra is a factor if its centre is trivial, i.e. a multiple of the identity. Every von Neumann algebra decomposes as a direct integral of factors. Two von Neumann algebras are algebraically isomorphic if there exists an invertible linear continuous map that preserves the *-algebraic structure. An element of the dual of a von Neumann algebra is usually called a state or a weight on the algebra. It is called positive if it maps positive operators (i.e operators whose spectrum lies in $[0,\infty[$) into the positive numbers. A representation of a von Neumann algebra is a linear *-homomorphism into the algebra of bounded operators on a Hilbert space. Every positive weight, w, on a von Neumann algebra defines a representation of the algebra via the celebrated GNS representation. The GNS representation defines the scalar product on the Hilbert space of the representation as $\langle A \mid B\rangle = w(A^*B)$ and the action of the algebra on the Hilbert space in the obvious way.

The basic result in the classification of von Neumann algebras is the existence of an algebraic invariant of the algebra, the von Neumann dimension function, that associates to every orthogonal projector, P, of the algebra ($P = P^*, P = P^2$) a non-negative number. Factors are classified according to the range of values that the dimension function may assume:

- type I: $0, 1, 2, \ldots \infty$
- type II_1: $[0, 1]$
- type II_∞: $[0, \infty]$
- type III: $0, \infty$.

A von Neumann algebra is called finite if it is finite the range of its dimension function. The theory of von Neumann algebras is a non-commutative far reaching generalisation of the integration theory and spectral theorem to which it reduces in the commutative case. Indeed every commutative von Neumann algebra is algebraically isomorphic to the algebra of all bounded measurable functions $L^\infty(X, d\mu)$ for some measure space X and thus every weight on a commutative von Neumann algebra is a measure.

One of the most striking features of the von Neumann algebras is the occurrence of a continuous dimension. In fact for type I and type II the dimension function can be identified with the trace (suitably normalised) of an orthogonal projection. Therefore type I and II possess semi-finite traces, while type III has no non-trivial trace. A trace, τ, is an element of the dual of the algebra that satisfies $\tau(AB) = \tau(BA)$. If it exist it is unique for a factor up to a multiplicative constant.

After years of efforts since their invention by Murray and von Neumann a general structure theorem has been achieved in the classification of the hyper-finite (or amenable or injective) factors with separable dual. A von Neumann algebra is hyper-finite if it is the closure of a sequence of finite dimensional matrix algebras. Though many von Neumann algebras satisfy this property, there are some interesting algebras that do not, as the Cuntz algebra with more than one self-adjoint generator. Here it is the complete list of the hyper-finite factors with separable dual.

- I_n: the algebra of matrices of rank n.
- I_∞: the algebra of all bounded operators on a Hilbert space with countable basis.
- R: the unique hyper-finite factor of type II_1. R has many equivalent realizations: as the countable tensor product of $n \times n$ matrices equipped with the corresponding unique factorized normalised trace; as a tracial representation of the canonical anti-commutation relations on an infinite dimensional euclidean space; as a Krieger factor associated to an ergodic transformation with an invariant measure.
- $R_{0,1} = R \times I_\infty$: the unique hyper-finite factor of type II_∞.
- III_0: R_W, the Krieger factor associated to an ergodic flow with a quasi-invariant measure.
- III_λ: R_λ with $\lambda \neq 0, 1$, the Powers factors.
- III_1: $R_\infty = R_{\lambda_1} \times R_{\lambda_2}$ with λ_1/λ_2 not rational, the algebra of canonical commutation relations in infinite dimensions.

In particular the Powers construction of hyper-finite factors of all types in terms of pairs: (infinite inductive tensor product of matrix algebras, factorized state over the tensor product), whose type depends on certain conditions about the eigenvalue list of the states, turned out to be fundamental.

We now discuss an application of the algebraic approach to the large-N limit that had some relevance for us in the introduction and in section 3. The simplest example of a hyper-finite master algebra is the one generated by functions of any single self-adjoint operator, that obviously is a commutative one, equipped with an arbitrary tracial state. This may be the algebra of local observables of a one-matrix model or, less trivially, the sub-algebra generated by the master field for a self-avoiding loop of area A in the large N-limit of QCD_2:

$$\exp\left(i\sqrt{A}(a + a^*)\right), \tag{A.1}$$

where a and a^* are elements of the Cuntz algebra. In this commutative case hyper-finiteness follows from the Szego limit theorem for self-adjoint operators, that depends essentially on the fact that abelian C^*-algebras are type I and thus every trace is uniformly approximately finite dimensional. In fact the interested reader may find in [18] a detailed study of how the approximation properties of traces carry information about (tracial) representation theory of von Neumann algebras.

B. Hitchin systems

Hitchin has studied a non-canonical reduction of the four dimensional self-dual equations:

$$F_{\mu\nu} = ({}^*F)_{\mu\nu} \tag{B.1}$$

to two dimensions, requiring that the four dimensional gauge connection be constant along two of the four coordinates [5].

The reduction is non-canonical (i.e. depending explicitly on a choice of the coordinate system) since Hitchin identifies the components of two polarisations of the four-dimensional connection $(A_u, A_{\bar{u}})$ with the components of a two dimensional one-form $(\Psi = \psi + \bar{\psi} = A_u dz + A_{\bar{u}} d\bar{z})$, the Higgs field, in order to write the reduced self-duality equations:

$$\begin{aligned} F_A - i\Psi^2 &= 0 \\ \bar{\partial}_A \psi &= 0 \\ \partial_A \bar{\psi} &= 0 \end{aligned} \tag{B.2}$$

as a flat condition for the non-hermitean connection $B = A + i\Psi$:

$$F_B = 0 \tag{B.3}$$

plus a harmonic condition:

$$d_A^* \Psi = 0 \tag{B.4}$$

for the metric g associated to the one form Ψ [78]:

$$\Psi = g^{-1} dg\,. \tag{B.5}$$

Hence the name of harmonic bundles for the Hitchin systems. The non-canonical nature of the Hitchin reduction matches nicely with the use that we have made of it in the partially quenched or twisted EK models in section 3. Hitchin has interpreted the reduced self-duality equations as the three moment maps of a hyper-Kahler reduction for the hamiltonian action of the local gauge group over the cotangent bundle of unitary connections (here we take the gauge group to be SU(N) or U(N)), labelled by the pairs (A, Ψ). Hitchin equations can be generalised to include coadjoint orbits at points p of a divisor D:

$$\begin{aligned} F_A - i\Psi^2 &= \sum_p \mu_p^0 \delta^{(2)}(x - x_p) \\ \bar{\partial}_A \psi &= \sum_p n_p \delta^{(2)}(x - x_p) \\ \partial_A \bar{\psi} &= \sum_p \bar{n}_p \delta^{(2)}(x - x_p)\,. \end{aligned} \tag{B.6}$$

It is known that for fixed eigenvalues of the hermitean levels, μ_p^0, the moduli space corresponding to taking the quotient by the action of the local compact gauge group is hyper-Kahler for n_p nilpotent. Alternatively the moduli space is hyper-Kahler for fixed coadjoint orbits, i.e. fixed monodromy around the punctures [69].

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From our point of view, including coadjoint orbits on an arbitrary divisor is needed to provide a stratification of the functional integral that be, at the same time, dense in the sense of distributions and arising from geometry. In fact only for a collective field of the form:

$$\begin{aligned}\mu^0 &= \sum_p \mu_p^0 \delta^{(2)}(x - x_p) \\ n &= \sum_p n_p \delta^{(2)}(x - x_p) \\ \bar{n} &= \sum_p \bar{n}_p \delta^{(2)}(x - x_p)\end{aligned} \tag{B.7}$$

existence of solutions for the microcanonical ensemble of section 4 with a nice (i.e. Hausdorff) moduli space are guaranteed.

The corresponding equation for B now reads:

$$F_B = \mu^0 + n - \bar{n} \tag{B.8}$$

together with the harmonic condition:

$$d_A^* \Psi = n + \bar{n} = \nu\,. \tag{B.9}$$

The harmonic bundles defined by the moment map equations, with coadjoint orbits included, are called tame harmonic bundles since in this case the eigenvalues of ψ are multi-valued holomorphic functions with poles at most of order one at the divisor of the coadjoint orbits. There are correspondences between tame harmonic bundles and several other objects of algebraic and topological origin [65]–[67]. Some of these correspondences play a fundamental role in this paper because they allow us to give a precise meaning, on the dense set defined by Hitchin systems, to the formal change of variables that occur in the derivation of the master equation of section 5 from the loop equations. One way of understanding the existence of such correspondences comes from the following general result about hyper-Kahler quotients. As we have seen the hyper-Kahler quotient is obtained imposing the three moment maps and taking the quotient by the action of the compact gauge group. In this way we get the moduli space of tame harmonic bundles. However we can choose one of the three complex structures and the associated non-hermitean moment map and take the quotient under the action of the complexification of the local gauge group, getting the same moduli space. From eq. (B.8) it is follows that to a tame harmonic bundle corresponds a flat connection and thus a local system, i.e. a representation of the fundamental group of the surface with a divisor. More precisely a local system is a fibre bundle with locally constant transition function and sections v given by solutions of the equations $d_B v = 0$ for the flat connection B. The moduli of local systems can be written in terms of the monodromy of the connection B along the generators of the fundamental group of the Riemann surface with a divisor, subjected to the consistency condition:

$$\Psi(a)\Psi(b)\Psi(a)^{-1}\Psi(b)^{-1}\prod_p \Psi(D_p) = 1 \tag{B.10}$$

after taking the quotient by the action of the global gauge group that acts by conjugation on the monodromies [75]. In addition to the flat connection it is possible to associate a flat holomorphic connection with regular singularities at the punctures:

$$\bar{\partial} b' = i\mu' \,. \tag{B.11}$$

This is the holomorphic de Rham local system we refer to in the paper. Taking a reduction with respect to another complex structure we get the Dolbeault moduli space of Higgs bundles defined by:

$$\bar{\partial}_A \psi = n \,. \tag{B.12}$$

The Dolbeault moduli do not play any role in section 5 but they will in section 6 when we consider a dual interpretation of the master equation based on the quantisation of Hitchin systems. In particular Hitchin hamiltonians are polynomials in ψ:

$$H_i = \mathrm{Tr}(\psi^i) \,. \tag{B.13}$$

For completeness we report a formula for the dimension of the moduli spaces of $\mathrm{SU}(N)$ Hitchin bundles on a Riemann surfaces with a divisor and nilpotent residue [79]–[83].

$$\mathrm{dim} N = (2g-2)(N^2-1) + \frac{1}{2} \sum_{p \in D} f_p \,, \tag{B.14}$$

where $f_p = N^2 - \sum_p m_p^2$ and m_p the multiplicity of the weights of the parabolic flag at p [82]. We have seen in section 5 that for physical purposes, i.e. for calculating the string tension, it is more convenient to compute the unitary monodromy $\Psi(x, x; A)$ on the master field rather than the B monodromy. We need therefore a way of separating the field A from Ψ in the loop equations. The natural way of doing it is to consider the new connection $B^\kappa = A + i(\kappa\psi + \bar{\kappa}\bar{\psi})$ obtained by a suitable rescaling of the field ψ. The A monodromy can then be recovered in principle in the limit $\kappa \to 0$. We can thus compute the effective action for the microcanonical ensemble appropriate for B^κ and try to solve the master equation in the limit $\kappa \to 0$. However we can do better. As it has been commented in the main text, it is in the spirit, if not in the letter of the master field idea, that the master field should be a κ independent solution of the master equation up to gauge equivalence. Thus A and Ψ should be fixed points for the κ action up to gauge equivalence. In finite dimensions it is known that such fixed points exist and are precisely those Hitchin bundles that arise as variations of Hodge structures [65]–[68]. In addition for such finite dimensional bundles the monodromy of B^κ is κ independent and coincides with $\Psi(x, x; A)$ at the fixed point [68].

C. Loop equations for QCD_2

In this appendix we study the loop equations for self-avoiding loops in large-N QCD_2 and we compare them with non-abelian localisation.

Localisation was discovered by Duistermaat and Heckmann [38] as a mean to compute exactly certain integrals by the saddle-point method despite they may not be gaussian.

It turned out that what was really involved in such integrals was some sort of cohomology. An infinite dimensional version of DH localisation due to Bismut [39, 40] was employed by Witten [37] for computing exactly the partition function and the cohomology ring of QCD_2.

For the sphere Witten localisation reduces to the saddle-point method. Since it depends on equivariant cohomology, Witten localisation applies to the partition function and to the observables belonging to the cohomology ring, but it does not apply to all the gauge invariant observables of QCD_2. In particular it does not apply to Wilson loops. Nevertheless it turns out that for self-avoiding loops on a large sphere the result derived by large-N functional techniques and the one obtained were to hold localisation are surprisingly similar though they do not agree exactly.

Indeed there is no reason why they should agree exactly since there is no reason why localisation should apply to Wilson loops. Here we use the techniques of section 3 and section 4 to explain where the discrepancy may arise.

We have seen that our methods for loop equations lead to a result similar to localisation, in the sense that they transform loop equations for self-avoiding loops into a saddle-point problem for a certain effective action. Yet our saddle-point problem is in fact defined over a U(1) extension of the original theory, since as a consequence of the loop equations a central extension is present in the master equation. The functional techniques and two dimensional localisation instead would predict that there is no central extension at all. However we observed in the main text that, despite the master equation is a priori defined over a central extension, we should require that the central extension vanishes due to a counter-term in the chiral jacobian of the change of variables, as it actually happens provided we perform a finite conformal rescaling of the metric. In addition we showed that the master equation in our lattice or adelic version depends on the (regularized) value of the derivative $\frac{\partial t}{\partial z}(p)$ of the uniformization map at the base-point p of the loop, identified as a puncture of the bordered punctured surface internal to the loop. For QCD_2 our master equation coincides with the saddle-point equation obtained by localisation, up to the rescaling due to the mentioned derivative of the uniformization map.

Thus it is precisely the existence of the uniformization map the source of the discrepancy between DH localisation in $d = 2$ and the present approach. Indeed we should remind the reader that the classical action is written in terms of fields living on the space-time manifold with global coordinates $(z, \bar{z})$, while the functional determinants that arise in the effective action are defined in terms of fields living on the image of the uniformization map, with coordinates $(t, \bar{t})$. Unfortunately we do not have a detailed information about the uniformization map, that by the way is singular at punctures, so that it must be regularized and suitably interpreted. What we do here is to find the value of $\frac{\partial t}{\partial z}(p)$ that is implied by the exact result obtained by functional methods. We find $|\frac{\partial t}{\partial z}(p)|^2 = N_C$, where N_C is the number of punctures inside the loop. Localisation would imply instead $|\frac{\partial t}{\partial z}(p)|^2 = 1$, somehow incorrectly unless the loop contains just one lattice point.

Let us present the detailed computation. For QCD_2 it easy to follow the steps that lead to the master equation for QCD_4. The microcanonical ensemble is described by the equation:

$$F_A - \mu = 0 \tag{C.1}$$

that is the moment map at level μ for the hamiltonian action of the local gauge group with respect to the symplectic structure:

$$\omega = \int d^2x \, \mathrm{Tr}(\delta A_z \wedge \delta A_{\bar{z}}) \, . \tag{C.2}$$

Alternatively eq. (C.1) is the zero level condition for the gauge action on the gauge connection and the coadjoint orbits. Inserting the resolution of the identity:

$$1 = \int \delta(F_A - \mu) D\mu \tag{C.3}$$

into the functional integral:

$$0 = \int DAD\mu \exp\left(-\frac{N}{2\lambda}\int \mathrm{Tr}\, F_A^2 d^2x\right)\delta(F_A - \mu) \tag{C.4}$$

and changing variables in the functional integral from the moduli field μ of the unitary connections to the moduli μ' of the corresponding holomorphic local systems:

$$\bar{\partial} A'_z - i\mu' = 0 \tag{C.5}$$

we get the effective action Γ for the moduli field μ':

$$0 = \int D\mu' \exp(-\Gamma) \, , \tag{C.6}$$

where Γ is given by:

$$\Gamma = \frac{N}{2\lambda}\int \mathrm{Tr}\, F_A^2 d^2x - \log \mathrm{Det}'(d_A \wedge) + \log \mathrm{Det}'\left(\frac{D\mu}{D\mu'}\right) \tag{C.7}$$

with the first determinant arising from localisation on a given level and the second determinant from the chiral change of variables (we assume that we are on a sphere or on a torus so that there are no non-local moduli). Here the label $'$ for the functional determinants means that the zero modes coming from gauge invariance must be projected away. Then, as in section 4, the loop equations follow in the form:

$$\begin{aligned} 0 = \int D\mu' \exp(-\Gamma)\Bigg(&\mathrm{Tr}\left(\frac{\delta\Gamma}{\delta\mu(w)}\Psi(x,x;A)\right) + \\ &+ \int_{C(x,x)} dy_z \bar{\partial}^{-1}(w-y)\, \mathrm{Tr}(\Psi(x,y;A))\, \mathrm{Tr}(\Psi(y,x;A))\Bigg) \, . \end{aligned} \tag{C.8}$$

From which we get the master equation:

$$0 = \frac{\delta\Gamma}{\delta\mu(x)} - \frac{i}{2}\mathrm{Ind}_C(x)\, \mathrm{Tr}(1) \tag{C.9}$$

Γ_{eff} can be computed non-formally by adding the covariant gauge-fixing:

$$d_A^* \delta A = 0 \tag{C.10}$$

with the corresponding Faddeev-Popov determinant and a choice of the metric:

$$\begin{aligned}\Gamma_{\text{eff}} = & \frac{N}{2\lambda} \int \operatorname{Tr} F_A^2 d^2x - \frac{1}{2} \log \operatorname{Det}(-\Delta_A 1 + i ad_{F_A}) + \frac{1}{2} \log \operatorname{Det}(d_A^* d_A) + \\ & + \log \operatorname{Det} \left(\frac{D\mu}{D\mu'} \right) - \epsilon(C) \frac{i}{2} \int \operatorname{Tr} \mu d^2x \,.\end{aligned} \tag{C.11}$$

However for computational purposes it is more convenient a different choice of gauge. If our Riemann surface is a sphere, the Hitchin system lives over a punctured sphere with punctures forming a uniform lattice (i.e. rotational invariant). The universal cover of a punctured sphere is the upper half plane H. Our punctured sphere is then conformally equivalent to a polygon in H with cusps on the x-axis corresponding to the punctures. On H we choose the gauge $A_y = 0$, that leaves, as a residual gauge symmetry, gauge transformations that are y independent. In the gauge $A_y = 0$ the determinant due to localisation and the Faddeev-Popov determinant are both field independent and cancel each other. We can use the residual gauge symmetry to fix the gauge $\mu^{ch} = 0$ (the label ch means the non-diagonal part) at the cusps (i.e the punctures) on the x-axis. Then the effective action reduces to:

$$\begin{aligned}\Gamma_{\text{eff}} = & \frac{N}{2\lambda a^2} \sum_i \sum_p h_p^{(t)i2} \left| \frac{\partial t}{\partial z}(p) \right|^2 + \sum_{i \neq j} \sum_p \log \left(h_p^{(t)i} - h_p^{(t)j} \right) + \\ & + \log \operatorname{Det} \left(\frac{Dh^{(t)}}{Dh^{(t)'}} \right) - \epsilon(C) \frac{i}{2} \int \operatorname{Tr} h^{(t)} d^2t \,,\end{aligned} \tag{C.12}$$

where the $^{(z)}$ and $^{(t)}$ superscript refer to the domain of definition of the lattice field, h_p, and we have set $a = \frac{2\pi}{\Lambda}$, with a the lattice spacing corresponding to the cutoff Λ of the theory, that comes from the product of delta functions at the same point in the classical action. The last two terms in eq. (C.12) cancel each other (modulo 1) (up perhaps to terms of higher order in $h^{(t)}$) provided the conformal factor in the metric is rescaled by a factor of π (see section 5). The result coming from localisation would be instead:

$$\Gamma = \frac{N}{2\lambda a^2} \sum_p h_p^{(z)i2} + \sum_{i \neq j} \sum_p \log(h_p^{(z)i} - h_p^{(z)j}) \tag{C.13}$$

that is similar up to the mentioned discrepancy in the field definitions. Should we compute the Wilson loop using the master field obtained by Γ rather than Γ_{eff} we would get the wrong result because of the following subtlety. The critical equation for Γ is just the saddle-point equation for the eigenvalues in the $(z, \bar{z})$ coordinates. The master equation for Γ_{eff} is the critical equation for the lattice or adelic field of residues in the coordinates defined by the uniformization map, i.e. those labelled by t, after a conformal rescaling of a factor of π to cancel the central term in the effective action. Let us suppose that we compute the Wilson loop according to the naive cohomological localisation:

$$W(A) = Z^{-1} \int \exp\left(-\frac{N}{2\lambda a^2}\sum_p h_p^{i2} - \sum_{i\neq j}\sum_p \log\left(h_p^i - h_p^j\right)\right)\sum_i \exp\left(i\sum_p h_p^i\right). \quad \text{(C.14)}$$

Assuming translational invariance in the large-N limit we get:

$$W(A) = Z^{-1} \int \exp\left(-\frac{NN_D}{2\lambda a^2}\sum_i h^{i2} - \sum_{i\neq j} N_D \log(h^i - h^j)\right)\sum_i \exp\left(iN_C h^i\right), \quad \text{(C.15)}$$

where N_D is the number of punctures in our space-time manifold and N_C the number of punctures internal to our loop C. By changing variables to $\theta^i = N_C h^i$ we obtain:

$$W(A) = Z^{-1} \int \exp\left(-\frac{NN_D}{2\lambda (N_C a)^2}\sum_i \theta^{i2} - \sum_{i\neq j} N_D \log(\theta^i - \theta^j)\right)\sum_i \exp\left(i\theta^i\right). \quad \text{(C.16)}$$

The corresponding master equation for θ is:

$$\frac{N}{2\lambda N_C^2 a^2}\theta^i = \sum_{j;j\neq i}\frac{1}{\theta^i - \theta^j}. \quad \text{(C.17)}$$

This equation should be compared with eq. (2.16) that is the correct result obtained by functional methods:

$$\frac{N}{2\lambda A}\theta^i = \sum_{j;j\neq i}\frac{1}{\theta^i - \theta^j}. \quad \text{(C.18)}$$

Since $A = N_C a^2$ we see that eq. (C.17) predicts a string tension wrong by a factor of N_C.

In our master equation instead the classical term is rescaled by the factor of $|\frac{\partial t}{\partial z}(p)|^2$ while the functional determinants in the effective action are not. The master equation derived by Γ_{eff} now reads:

$$\frac{N|\frac{\partial t}{\partial z}(p)|^2}{2\lambda\pi a^2}h_p^{(t)i} = \sum_{j;j\neq i}\frac{1}{h_p^{(t)i} - h_p^{(t)j}}, \quad \text{(C.19)}$$

where we have included the conformal rescaling of a factor of π. The equation for $\theta^i = N_C h_p^i$ is then:

$$\frac{N|\frac{\partial t}{\partial z}(p)|^2}{2\pi\lambda N_C a^2}\theta^i = \sum_{j;j\neq i}\frac{N_C}{\theta^i - \theta^j} \quad \text{(C.20)}$$

that is the correct one provided $|\frac{\partial t}{\partial z}(p)|^2 = N_C$. We can understand naively why $|\frac{\partial t}{\partial z}(p)|^2$ should be of order of N_C. Let us recall that our manifold has the topology of a punctured disk, with the boundary of the disk forming the "large horocycle", while the punctures are regularized by means of small horocycles around them. Following [64, 84], given a metric on a fatgraph with lengths l_i, there is a unique quadratic differential q such that the residue of $\sqrt{q}$ at the puncture is l_i, the length of the horocycle around it. We will suppose that this picture extends to the case that the puncture is on the "large horocycle". We can identify the differential $\sqrt{q}$ with $\frac{\partial t}{\partial z}(p)$. After regularization it is of order of $\frac{l_p}{a}$ that is proportional to the ratio between the lengths of the "large" and small horocycles. Its square is proportional to the ratio of the corresponding areas. But this is proportional to the number of punctures N_C inside the "large horocycle".

D. Asymptotic freedom

In this appendix we compute the divergent part of the localisation determinant. It is convenient to perform the computation in an indirect way, by means of a term by term comparison with the usual one-loop perturbative contribution to the effective action. For this purpose let us recall the one-loop perturbative result that is given by:

$$\begin{aligned}&\int DcDA_\mu \exp\left(-\frac{N}{2\lambda}\operatorname{Tr}(c^2)\right)\exp\left(-\frac{N}{2\lambda}\operatorname{Tr}(F^2_{\mu\nu})\right)\delta(D_\mu\delta A_\mu - c)\Delta_{FP} = \\ &\quad = \operatorname{Det}^{-1/2}(-\Delta_A\delta_{\mu\nu}1 + 2iad_{F_{\mu\nu}})\operatorname{Det}(-\Delta_A)\,, \end{aligned} \tag{D.1}$$

where we have inserted the gauge-fixing condition and the corresponding Faddeev- Popov determinant and by an abuse of language we have denoted with A the classical background field in the right hand side of eq. (D.1). The perturbative computation of the one-loop beta function is the result of two contributions that are independent within logarithmic accuracy [57]. The orbital contribution gives rise to diamagnetism and to a positive term in the beta function:

$$\begin{aligned}\log\left(\operatorname{Det}^{-1/2}(-\Delta_A\delta_{\mu\nu}1)\operatorname{Det}(-\Delta_A)\right) &= \log\operatorname{Det}^{-1}(-\Delta_A) \\ &= \frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\operatorname{Tr}(F_{\mu\nu})^2\,,\end{aligned} \tag{D.2}$$

where it should be noticed the cancellation of two of the four polarisations between the first factor and the Faddev-Popov determinant. The spin contribution gives rise to paramagnetism and to a overwhelming negative term in the beta function [57]:

$$\frac{1}{4}\operatorname{Tr}\left(i2ad_{F_{\mu\nu}}(-\Delta_A)^{-1}i2ad_{F_{\mu\nu}}(-\Delta_A)^{-1}\right) = -\frac{12}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\operatorname{Tr}(F_{\mu\nu})^2\,. \tag{D.3}$$

Hence the complete result is:

$$-\frac{11}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\operatorname{Tr}(F_{\mu\nu})^2 = -\beta_1\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\operatorname{Tr}(F_{\mu\nu})^2\,. \tag{D.4}$$

In the case of the localisation determinant we have instead:

$$\begin{aligned}\operatorname{Det}^{-1/2}(-\Delta_1) &= \lim_{\epsilon\to 0}\int DcDA_\mu\exp\left(-\frac{1}{\epsilon}\operatorname{Tr}(c^2)\right)\times \\ &\quad\times\exp\left(-\frac{1}{\epsilon}\operatorname{Tr}((F^-_{\mu\nu}-\mu^-_{\mu\nu})^2)\right)\delta(D_\mu\delta A_\mu - c)\Delta_{FP}\,,\end{aligned} \tag{D.5}$$

where we have inserted the gauge-fixing condition and the corresponding Faddeev-Popov determinant and by an abuse of language we have denoted by A the background field defined by the condition $F^-_{\mu\nu} - \mu^-_{\mu\nu} = 0$. The result for $\mathrm{Det}^{-1/2}(-\Delta_1)$ is thus:

$$\mathrm{Det}^{-1/2}(-\Delta_1) = \mathrm{Det}^{-1/2}(-\Delta_A \delta_{\mu\nu} 1 + iad_{F^-_{\mu\nu}})\,\mathrm{Det}(-\Delta_A)\,. \tag{D.6}$$

The coefficient of the spin contribution in this case is one half of the one in the perturbative case, but the spin term involves the anti-selfdual part of the field strength instead of the field strength itself. The orbital contribution is the same as in the perturbative case. Thus we have for the orbital part:

$$\begin{aligned}\frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\,\mathrm{Tr}(F_{\mu\nu})^2 &= \frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2 + \\ &+\frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^+_{\mu\nu})^2\,,\end{aligned} \tag{D.7}$$

where we have used the decomposition:

$$F_{\mu\nu} = \frac{1}{2}F^-_{\mu\nu} + \frac{1}{2}F^+_{\mu\nu}\,. \tag{D.8}$$

For the spin part we get:

$$\frac{1}{4}\,\mathrm{Tr}\left(iad_{F^-_{\mu\nu}}(-\Delta_A)^{-1} iad_{F^-_{\mu\nu}}(-\Delta_A)^{-1}\right) = -\frac{12}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2\,. \tag{D.9}$$

We see that the spin part has the correct factor of 2 once it is written in terms of the anti-selfdual part of the field strength, a fact that we could have anticipated from start, to arrive at the final answer:

$$-\frac{11}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^-_{\mu\nu})^2 + \frac{1}{12}\frac{N}{(2\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\frac{1}{4}\,\mathrm{Tr}(F^+_{\mu\nu})^2 \tag{D.10}$$

that is the one reported is section 5.

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